Separating Markov's Principles

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Markov's Principle and Equivalent Sentences

Markov's Principle

Given a binary sequence $(b_i)_{i:\mathbb{N}}$, if not all of its entries are 0, then $b_i = 1$ for some i.

This is equivalent to:

- Post's theorem: A Σ^0_1 predicate whose complement is Σ^0_1 is decidable.
- "A computation halts if it does not loop."
- Completeness of Σ^0_1 theories of classical first-order logic with respect to Tarski models.

Object Type Theory

We model MLTT with:

- $\mathbb{B}$, $\mathbb{N}$, empty and unit types;
- Π and Σ types;
- A universe $\mathbb{U}$;
- A truncation $\| \cdot \|$ into propositions.

Markov's Principles in Type Theory

Depending on our reading of decidable predicate, we state MP in the object type theory as:

$$\mathsf{MP}_{\mathbb{U}} := \forall A : \mathbb{N} \to \mathbb{U}.\ (\forall n.\ An \vee \neg An) \to \qquad \neg\neg(\exists n.\ \|An\|) \to \exists n.\ \|An\|$$

$$\mathsf{MP}_{\mathbb{B}} := \forall f : \mathbb{N} \to \mathbb{B}. \qquad \neg\neg(\exists n.\ fn = \mathsf{true}) \to \exists n.\ fn = \mathsf{true}$$

$$\mathsf{MP}_{\mathsf{PR}} := \forall f : \mathbb{N} \to \mathbb{B}.\ \mathsf{primitive\text{-}recursive}\, f \to \neg\neg(\exists n.\ fn = \mathsf{true}) \to \exists n.\ fn = \mathsf{true}$$

Some Implications

$$\mathsf{MP}_{\mathbb{U}} := \forall A : \mathbb{N} \to \mathbb{U}.\ (\forall n.\ An \lor \neg An) \to \quad \neg\neg(\exists n.\ \|An\|) \to \exists n.\ \|An\|$$

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$$\mathsf{MP}_{\mathbb{U}} \longrightarrow \mathsf{MP}_{\mathbb{B}} \longrightarrow \mathsf{MP}_{\mathsf{PR}}$$

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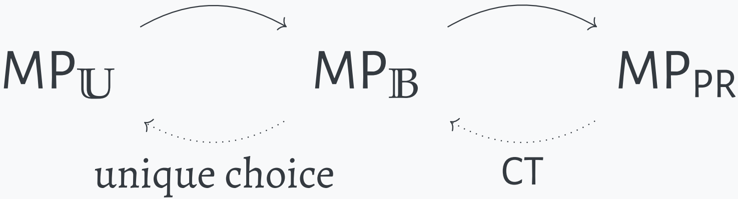


Realizability Models of MLTT with Choice Sequences

Use two different instantiations of $\mathrm{TT}^{\square}_{\mathcal{C}}$ to get the following models:

First model of MLTT

Choice sequences of Booleans

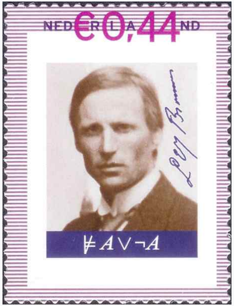
$\mathrm{MP}_{\mathrm{PR}}$	✓
$\mathrm{MP}_{\mathbb{B}}$	✗
$\mathrm{MP}_{\mathbb{U}}$	✗

Second model of MLTT

Choice sequences of propositions

$\mathrm{MP}_{\mathrm{PR}}$	✓
$\mathrm{MP}_{\mathbb{B}}$	✓
$\mathrm{MP}_{\mathbb{U}}$	✗

Brouwer's Choice Sequences



- Infinite sequences whose values are "generated" over time.
- Only have access to a prefix of the sequence at any given time.

A Formal Reading of Choice Sequences

Fix a pre-ordered set $(\mathbb{W}, \sqsubseteq)$.

A (Boolean) choice sequence is a function $f : \mathbb{N} \to \mathbb{W} \to \mathbb{B}_\perp$ such that:

- for all n, $f(n)$ is monotonic
- for all n and paths $(w_i)_{i:\mathbb{N}}$ through $\mathbb{W}$, there exists some m such that $f(n)(w_m) \downarrow$

An Informal Formal Reading of Choice Sequences

Fix a pre-ordered set $(\mathbb{W}, \sqsubseteq)$.

A (Boolean) choice sequence is a function $f : \mathbb{N} \to \mathbb{W} \to \mathbb{B}_\perp$ such that:

- for all n, $f(n)$ is monotonic
  - once an entry is generated it cannot change
- for all n and paths $(w_i)_{i:\mathbb{N}}$ through $\mathbb{W}$, there exists some m such that $f(n)(w_m) \downarrow$
  - every entry will eventually be generated

Covering Relation

$TT^{\square}_{\mathcal{C}}$ can be instantiated with different covering relations, for this work we use:

Beth Covering

An upwards-closed subset $U \subset \mathbb{W}$ covers a world w if:

- for all paths $(w_i)_{i:\mathbb{N}}$ starting at w, we have some n with $w_n \in U$.

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(thanks Alex for the tutorial ☺)

First Separation Result

First model of MLTT

Choice sequences of Booleans

$\mathsf{MP}_{\mathsf{PR}}$	✓
$\mathsf{MP}_{\mathbb{B}}$	✗
$\mathsf{MP}_{\mathbb{U}}$	✗

First Separation: Proving $\mathsf{MP}_{\mathsf{PR}}$

The following rule is always derivable:

$$\frac{w \vDash \Gamma, n : \mathsf{Nat} \cap \mathsf{pure} \vdash \|Pn\|}{w \vDash \Gamma \vdash \forall n : \mathsf{Nat}.\|Pn\|}$$

Primite-recursive functions are encoded by elements of Nat, giving:

$$\mathsf{MP} \text{ in the metatheory} \implies \mathsf{MP}_{\mathsf{PR}} \text{ in the model}$$

First Separation: Disproving $\mathrm{MP}_{\mathbb{B}}$

The semantics of negation are as follows

$$w \vDash \neg A \iff \text{for all extensions } u \sqsupseteq w, u \nvDash A$$

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so in particular we have

$$w \vDash \neg\neg A \iff \text{for all extensions } u \sqsupseteq w, \text{there exists a further extension } v \sqsupseteq u, v \vDash A$$

First Separation: Disproving $MP_{\mathbb{B}}$

For any world w, pick an empty choice sequence α. We can always prove that

$$w \vDash \neg\neg(\exists n.\ \alpha(n) = \mathsf{true})$$

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requires us to show that across all paths at some point a true entry is generated. But there exists a path where only false entries are generated.

Hence we can negate $\mathsf{MP}_{\mathbb{B}}$.

Second Separation Result

Second model of MLTT

Choice sequences of propositions

$\mathsf{MP}_{\mathsf{PR}}$	✓
$\mathsf{MP}_{\mathbb{B}}$	✓
$\mathsf{MP}_{\mathbb{U}}$	✗

- Choice sequences of propositions don't allow defining any more functions $\mathbb{N} \to \mathbb{B}$
  $\rightsquigarrow$ MP in the metatheory $\implies$ $\mathsf{MP}_{\mathbb{B}}$ in the model
- Similar argument as before proves negation of $\mathsf{MP}_{\mathbb{U}}$

Conclusion

- Be careful how you state Markov's Principle!
- Choice sequences are great at falsifying classical principles concerning sequences.
  $\rightsquigarrow$ same setup works for different versions of LPO, and likely for WLPO and LLPO too.
- Realizability models allow for fine control over the allowed choice sequences.
  $\rightsquigarrow$ namely having boolean versus propositional choice sequences.

A Truncation Into Propositions

Homotopy Type Theorists Be Advised

We do not model the HoTT-style propositional truncation.

We are missing the typical universal mapping property

$$\Pi P : \Omega, (A \to P) \simeq (\|A\| \to P)$$

Instead we validate the following

$$\Pi B : \mathbb{U}, (A \to \|B\|) \simeq (\|A\| \to \|B\|)$$