Limited Principles of Omniscience in Constructive Type Theory

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“The” Limited Principle of Omniscience

Limited Principle of Omniscience

For all binary sequences $(b_i)_{i:\mathbb{N}}$, the proposition $\exists i : \mathbb{N}, b_i = 1$ is decidable.

LPO is strictly weaker than LEM over an intuitionistic base theory.

Of interest in reverse constructive mathematics.

Object Type Theory

We model MLTT with:

- $\mathbb{B}$, $\mathbb{N}$, empty and unit types
- Π and Σ types
- A universe $\mathbb{U}$
- A truncation $\| \cdot \|$ into propositions

A Truncation Into Propositions

Homotopy Type Theorists Be Advised

We do not model the HoTT-style propositional truncation.

We are missing the typical universal mapping property

$$\Pi P : \Omega, (A \to P) \simeq (\|A\| \to P)$$

Instead we validate the following

$$\Pi B : \mathbb{U}, (A \to \|B\|) \simeq (\|A\| \to \|B\|)$$

Limited Principles of Omniscience in Type Theory

Finally, we state LPO in the object type theory:

$$\mathsf{LPO}_{\mathbb{U}} := \forall A : \mathbb{N} \to \mathbb{U}.\ (\forall n.\ An \vee \neg An) \to (\exists n.\ \|An\|) \vee \neg(\exists n.\ \|An\|)$$

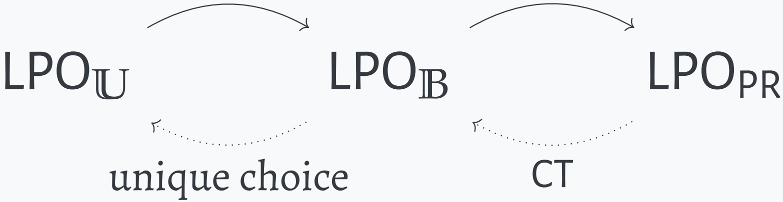
$$\mathsf{LPO}_{\mathbb{B}} := \forall f : \mathbb{N} \to \mathbb{B}.\ (\exists n.\ fn = \mathsf{true}) \vee \neg(\exists n.\ fn = \mathsf{true})$$

$$\mathsf{LPO}_{\mathsf{PR}} := \forall f : \mathbb{N} \to \mathbb{B}.\ \mathsf{primitive\text{-}recursive}\, f \to (\exists n.\ fn = \mathsf{true}) \vee \neg(\exists n.\ fn = \mathsf{true})$$

Some Implications

$$\mathrm{LPO}_{\mathbb{U}} \longrightarrow \mathrm{LPO}_{\mathbb{B}} \longrightarrow \mathrm{LPO}_{\mathrm{PR}}$$

Some Implications



Realizability Models with Choice Sequences

Use two different instantiations of $TT_{\mathcal{C}}^{\square}$ to get the following models:

First model

Choice sequences of Booleans

$\mathsf{LPO}_{\mathsf{PR}}$	✓
$\mathsf{LPO}_{\mathbb{B}}$	✗
$\mathsf{LPO}_{\mathbb{U}}$	✗

Second model

Choice sequences of propositions

$\mathsf{LPO}_{\mathsf{PR}}$	✓
$\mathsf{LPO}_{\mathbb{B}}$	✓
$\mathsf{LPO}_{\mathbb{U}}$	✗

Brouwer’s Choice Sequences

Infinite sequences whose values are “generated” with time.

Only have access to a finite prefix of the sequence.

A Formal Reading of Choice Sequences

Fix a pre-ordered set $(\mathbb{W}, \sqsubseteq)$.

A (Boolean) choice sequence is a function $f : \mathbb{N} \to \mathbb{W} \to \mathbb{B}_\perp$ such that:

- for all n, $f(n)$ is monotonic
- for all n and paths $(w_i)_{i:\mathbb{N}}$ through $\mathbb{W}$, there exists some m such that $f(n)(w_m) \downarrow$

An Informal Formal Reading of Choice Sequences

Fix a pre-ordered set $(\mathbb{W}, \sqsubseteq)$.

A (Boolean) choice sequence is a function $f : \mathbb{N} \to \mathbb{W} \to \mathbb{B}_\perp$ such that:

- for all n, $f(n)$ is monotonic
  - once an entry is generated it cannot change
- for all n and paths $(w_i)_{i:\mathbb{N}}$ through $\mathbb{W}$, there exists some m such that $f(n)(w_m) \downarrow$
  - every entry will eventually be generated

Covering Relation

$\mathrm{TT}^{\Box}_{\mathcal{C}}$ can be instantiated with different covering relations, for this work we use:

Beth Covering

An upwards-closed subset $U \subset \mathbb{W}$ covers a world w if:

- for all paths $(w_i)_{i:\mathbb{N}}$ starting at w, we have some n with $w_n \in U$.

Forcing Semantics

Some of the cases for the semantics:

$$w \vDash t_1 = t_2 \iff \exists U \text{ covering } w \text{ such that } \forall u \in U,\, t_i \downarrow^u \text{ and their values agree}$$

$$w \vDash A \wedge B \iff w \vDash A \text{ and } w \vDash B$$

$$w \vDash A \vee B \iff \exists U \text{ covering } w \text{ such that } \forall u \in U,\, u \vDash A \text{ or } u \vDash B$$

First Model: Proving $\mathsf{LPO}_{\mathsf{PR}}$

The following rule is always derivable:

$$\frac{w \vDash \Gamma, n : \mathsf{Nat} \cap \mathsf{pure} \vdash \ \|Pn\|}{w \vDash \Gamma \vdash \forall n : \mathsf{Nat}.\|Pn\|}$$

Primite-recursive functions are encoded by elements of Nat, giving:

$$\mathsf{LPO} \text{ in the metatheory} \implies \mathsf{LPO}_{\mathsf{PR}} \text{ in the model}$$

First Model: Disproving $\mathsf{LPO}_{\mathbb{B}}$

It is true that

$$\vDash \neg \mathsf{LPO}_{\mathbb{B}} \iff \text{for all worlds } w \nvDash \mathsf{LPO}_{\mathbb{B}}$$

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Fix w, assume $w \vDash \mathsf{LPO}_{\mathbb{B}}$ and instantiate with an empty choice sequence α giving:

$$w \vDash (\exists n.\ \alpha n = \mathsf{true}) \vee \neg(\exists n.\ \alpha n = \mathsf{true})$$

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We consider a path $(w_i)_{i:\mathbb{N}}$ where all entries of δ are generated as false. For some u in this path, we have either of :

$$u \vDash \exists n.\ \alpha n = \mathsf{true} \quad \text{or} \quad u \vDash \neg(\exists n.\ \alpha n = \mathsf{true})$$

First Model: Disproving $\mathsf{LPO}_{\mathbb{B}}$

Forcing conditions now take us to some extension $w \sqsubseteq u$ where all generated entries of α are false so far.

Suppose $u \vDash \exists n.\ \alpha n = \mathsf{true}$

or

Suppose $u \vDash \neg(\exists n.\ \alpha n = \mathsf{true})$

First Model: Disproving $LPO_{\mathbb{B}}$

Forcing conditions now take us to some extension $w \sqsubseteq u$ where all generated entries of α are false so far.

Suppose $u \vDash \exists n.\ \alpha n = \text{true}$

So $(\exists n.\ \alpha n = \text{true})$ becomes true along all paths from u.

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First Model: Disproving $\mathrm{LPO}_{\mathbb{B}}$

Forcing conditions now take us to some extension $w \sqsubseteq u$ where all generated entries of α are false so far.

Suppose $u \vDash \exists n.\ \alpha n = \mathsf{true}$

So $(\exists n.\ \alpha n = \mathsf{true})$ becomes true along all paths from u.

Pick path where entries are always generated as false, leading to $\lightning$

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Suppose $u \vDash \neg(\exists n.\ \alpha n = \text{true})$

So in all extensions $u \sqsubseteq v$ we have

$$v \nvDash \exists n.\ \alpha n = \text{true}$$

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So $\mathsf{LPO}_{\mathbb{B}}$ is false in our model.

Second Model: Proving $\mathrm{LPO}_{\mathbb{B}}$

Working with propositional choice sequences now:

Second Model: Proving $\mathrm{LPO}_{\mathbb{B}}$

Working with propositional choice sequences now:

1. Choice sequences of propositions don't allow defining any more functions $\mathbb{N} \to \mathbb{B}$
   - Proved using a simulation on closed terms

Second Model: Proving $\mathrm{LPO}_{\mathbb{B}}$

Working with propositional choice sequences now:

1. Choice sequences of propositions don't allow defining any more functions $\mathbb{N} \to \mathbb{B}$
   - Proved using a simulation on closed terms

2. Hence it suffices to consider pure functions giving:

$$\text{LPO in the metatheory} \implies \mathrm{LPO}_{\mathbb{B}} \text{ in the model}$$

Second Model: Disproving $\mathrm{LPO}_{\mathbb{U}}$

Working with propositional choice sequences now:

Second Model: Disproving $\mathrm{LPO}_{\mathbb{U}}$

Working with propositional choice sequences now:

1. Show that predicates arising from choice sequences are decidable

Second Model: Disproving $\mathrm{LPO}_{\mathbb{U}}$

Working with propositional choice sequences now:

1. Show that predicates arising from choice sequences are decidable
2. Continue with same argument as before

Conclusion

Choice sequences are great at falsifying classical principles concerning sequences.

Realizability models allow for fine control over the allowed choice sequences.