Timeful Mathematics

via Brouwer's Choice Sequences

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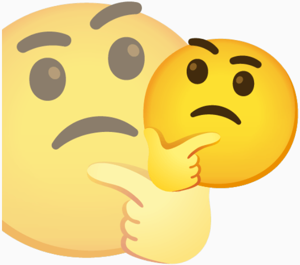
February 2024

The Path Ahead

1. Some quick philosophising on mathematics
2. Brouwer and his choice sequences
3. Using choice sequences to disprove Markov's Principle
4. How Brouwer's ideas are still relevant today

Nature of Mathematics



What is mathematics?

What is proof?

Who decides?

Platonism: Maths is Important

Mathematical Platonism

Mathematical objects are abstract entities that are eternal and unchanging.

Gödel was a major figure of mathematical platonism, believing that mathematics was an empirical science.

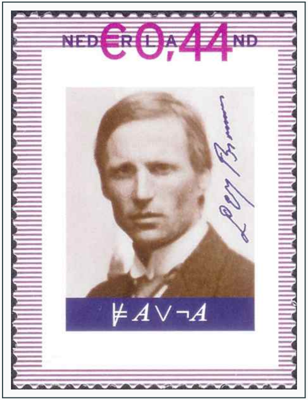
Formalism: Maths is Not Important

Formalism

Mathematics is the study of string manipulation according to rules.

Hilbert was a major figure with his program searching for a complete and consistent axiomatisation of mathematics.

Brouwer: a Contrarian



Born in the Netherlands in 1881, ended his doctoral studies in 1907.

Regarded as one of the founders of modern topology.

Also regarded as <u>very opinionated</u> on the nature of mathematics.

Brouwer's Intuitionism

Brouwer's Intuitionism

Mathematics is about mental constructions.

For the subject to prove a mathematical statement, the subject must perform an appropriate mental construction exhibiting it.

Introduces a temporal aspect to mathematics — what is true increases as the subject experiences more.

Choice Sequences by Example

Consider the following definition of an infinite sequence:

$$\alpha(n) = \begin{cases} 1 & \text{if P has been proven by time } n \\ -1 & \text{if P has been disproven by time } n \\ 0 & \text{otherwise} \end{cases}$$

At any point we only know a finite prefix of this sequence.

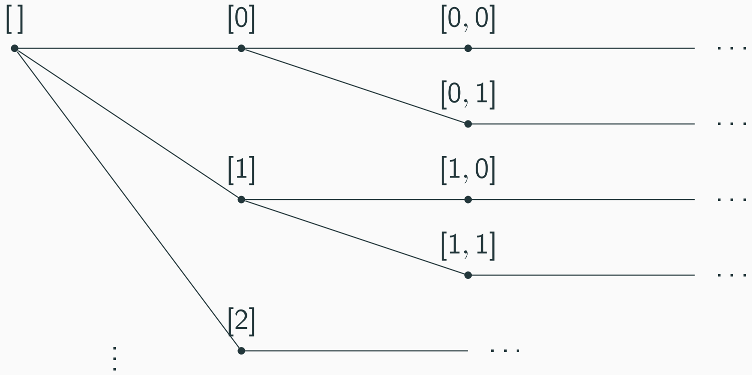
Choice Sequences by Definition

Choice Sequences

A choice sequence is an infinite sequence of which we only know of a finite prefix of.

Making Choice Sequences Formal

Consider the following set of worlds given by lists:



Making Choice Sequences Formal

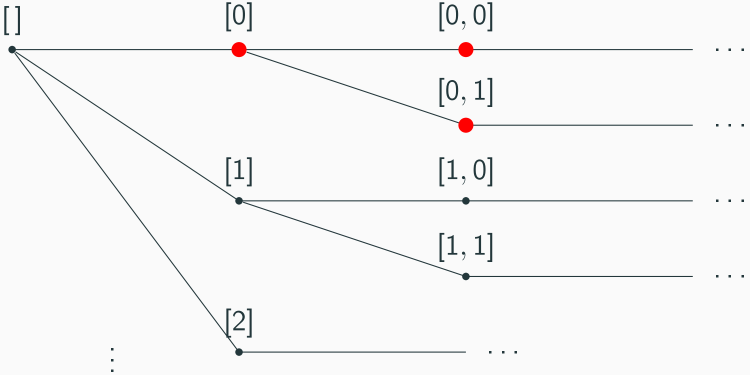
In relation to a world w, we evaluate $\alpha(n)$ as follows

$$\alpha(n) = \begin{cases} w[n] & \text{if } w \text{ has an nth entry} \\ \text{undefined} & \text{otherwise} \end{cases}$$

An equality is true if there exists some set of worlds covering all possible paths from the starting world where both sides are defined and the equalities between natural numbers hold.

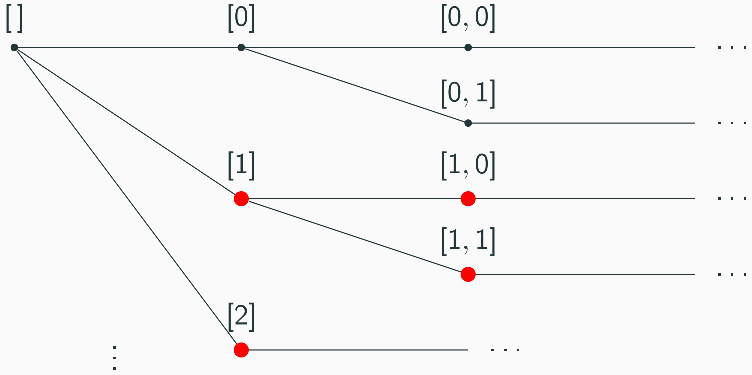
Understanding Equalities Involving Choice Sequences

The equality $\alpha(0) = 0$ is true:



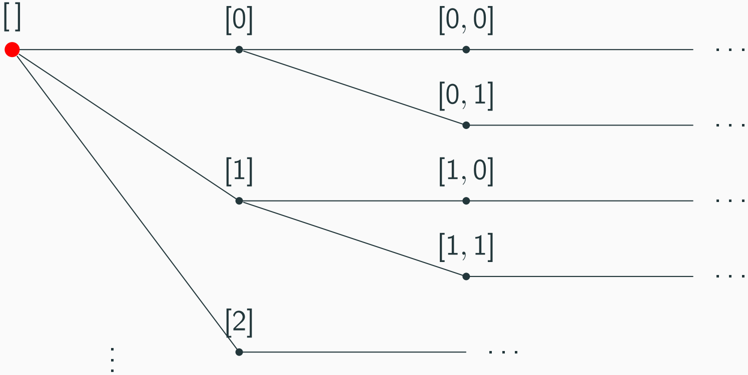
Understanding Equalities Involving Choice Sequences

The equality $\alpha(0) = 0$ is false:

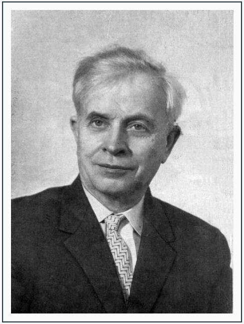


Understanding Equalities Involving Choice Sequences

The equality $\alpha(0) = 0$ is undecided:



Enter Markov's son, Markov Jr.



Born in Russia in 1903, finished his doctoral studies in 1924.

A prototypical computer scientist. Maybe just a computer scientist.

An Obvious Truth

Markov's Principle

For any decidable predicate A on natural numbers we have

$$(\neg\neg\exists n, A(n)) \implies (\exists n, A(n))$$

If it is impossible that no natural number exists satisfying A, then there must exist some natural number n such that $A(n)$ is true.

Justifying an Obvious Truth

```
def markov():
    n = 0
    while True:
        if checkA(n):
            break
        else:
            n = n + 1
    return n
```

Disproving an Obvious Truth

Proving statements about choice sequences is like playing a game against a belligerent crystal ball that always generates the most inconvenient entries.

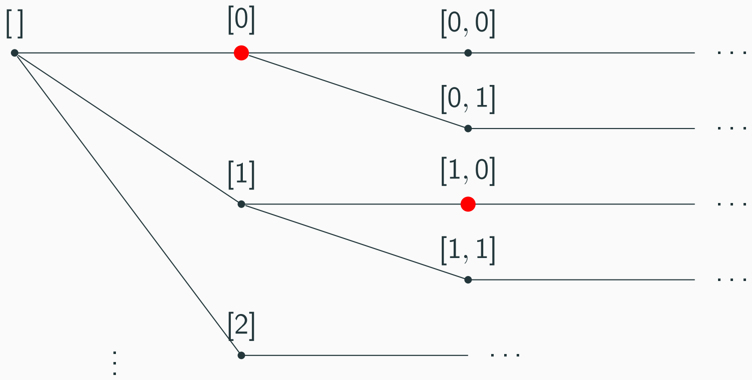
Double negations are easy to prove

To prove double negated statements such as

$$\neg\neg\exists n, \alpha(n) = 0$$

we only need one path where the crystal ball generates a 0.

Double negations are easy to prove



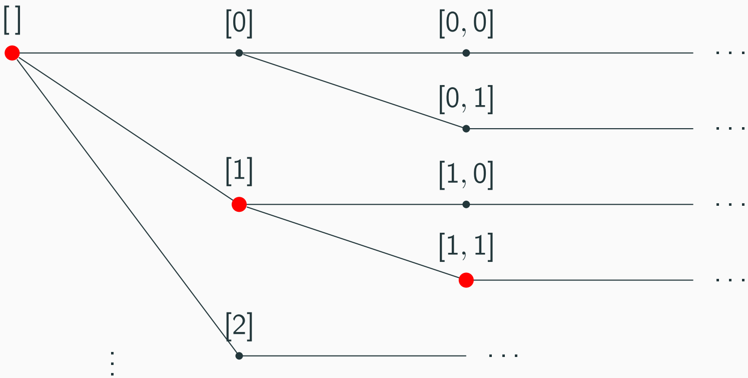
Non negations are hard to prove

To prove non-negated statements such as

$$\exists n, \alpha(n) = 0$$

we need the crystal ball to generate a 0 in all paths.

Non negations are hard to prove



Markov's Principle

For any decidable predicate A on natural numbers we have

$$(\neg\neg\exists n, A(n)) \implies (\exists n, A(n))$$

So we can disprove Markov's Principle instantiated with the predicate $\alpha(n) = 0$ for A.

The Current Day

By extending type theory with choice sequences, Coquand and Mannaa show that Markov's Principle is independent of type theory [5].

By varying the notion of choice sequences, different versions of Markov's Principle can be separated in type theory [7].

Brouwer's ideas on continuity and his bar thesis are also being explored in type theory [4, 6, 8, 1, 2, 3]

Conclusion

It is possible to envision mathematics as a timeful endeavour, as opposed to a timeless one.

Brouwer was a proponent of this view who used it with great effect to study a constructive account of analysis.

Even 100 years later, Brouwer's ideas are still relevant to the study of mathematical foundations and type theory.

References

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