# Forcing in Type Theory

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# Overview

- Kripke semantics for intuitionistic first order logic
- Beth semantics as a generalisation of Kripke semantics
- The internal logic of a (pre)sheaf topos
- Forcing constructive principles

# Kripke Models

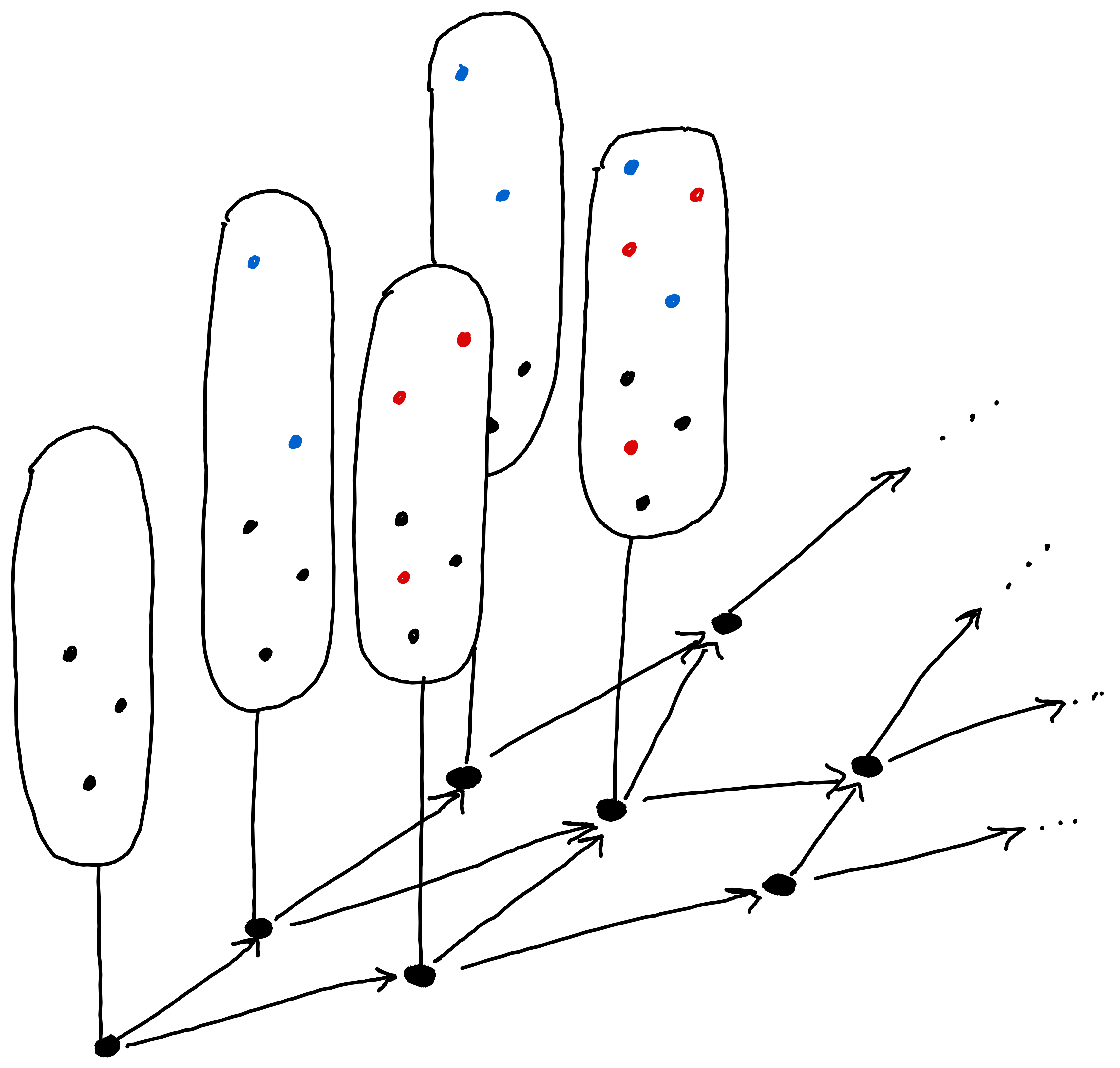
A Kripke Model for some relational language $\mathscr{L}$ consists of the following data

- A set $\mathbb{W}$ of possible worlds with a partial ordering $\leq$
- For each world $w \in \mathbb{W}$, a domain of discourse $M_w$
- For each world $w \in \mathbb{W}$, and relation $R \in \mathscr{L}$, some interpretation $R_w$ in $M_w$

Satisfying the following properties for any worlds such that $w \leq v$

- Their domains of discourse are related by $M_w \subseteq M_v$
- For any relation $R$ and elements $a_1, . . . , a_n \in M_w$, if $R_w\left(a_1, . . . , a_n\right)$ then $R_v\left(a_1, . . . , a_n\right)$

# Kripke Models Visualised



# Kripke Semantics

$w \Vdash_\sigma P\left(t_1, \ldots, t_n\right)$ $:= P_w\left([t_1]_\sigma, \ldots, [t_n]_\sigma\right)$

$w \Vdash_\sigma A \land B$ $:= w \Vdash_\sigma A$ **and** $w \Vdash_\sigma B$

$w \Vdash_\sigma A \lor B$ $:= w \Vdash_\sigma A$ **or** $w \Vdash_\sigma B$

$w \Vdash_\sigma \top$ $:=$ **true**

$w \Vdash_\sigma \bot$ $:=$ **false**

$w \Vdash_\sigma A \to B$ $:=$ **for all** $v \geq w$, $\left(v \Vdash_\sigma A\right)$ **implies** $\left(v \Vdash_\sigma B\right)$

$w \Vdash_\sigma \forall x \,.\, A$ $:=$ **for all** $v \geq w$ **and for all** $a \in M_v$, $v \Vdash_{\sigma, x \mapsto a} A$

$w \Vdash_\sigma \exists x \,.\, A$ $:=$ **there exists** $a \in M_v$, $w \Vdash_{\sigma, x \mapsto a} A$

# Consequences of Kripke Semantics

Kripke models are sound and complete with respect to intuitionistic first-order logic

**Monotonicity:**

For any formula $\phi$ and worlds $w \leq v$ in a Kripke model, if $w \Vdash_\sigma \phi$ then $v \Vdash_\sigma \phi$

**Local Character:**

For any formula $\phi$ and world $w$ in a Kripke model, if $v \Vdash_\sigma \phi$ for all extensions $v \geq w$ then $w \Vdash_\sigma \phi$

# Relaxing Kripke Semantics

Instead of asking for something to hold in all extensions, ask for it to eventually always hold

- Consider bars (ie. subsets) $U \subseteq \mathbb{W}$
  - Monotone bars (ie. upward closed subsets)
  - Decidable bars (ie. decidable subsets)
- Give meaning to “eventually always hold” through a covering relation $\lhd \subseteq \mathbb{W} \times \mathbf{Bars}$

By changing the notion of bars and the barring relation used, we can get different completeness results

# Beth Models

Classically these would look like Kripke models where the underlying poset is a tree.

In such a case, the notion of barring would be:

$$w \lhd U := \text{for all paths } \alpha \text{ starting at } w\text{, there exists some } n \in \mathbb{N} \text{ such that } \alpha(n) \in U$$

To avoid putting such requirements on the models we can work with a different notion of barring

$$w \lhd U := \text{for all } v \geq w,\ \text{there exists } u \geq v\text{, such that } u \in U$$

# Beth Semantics

$w \Vdash_\sigma P\left(t_1, \ldots, t_n\right)$ := **there exists a bar** $U$ **such that** $w \triangleleft U$ **and for all** $v \in U, P_v\left([t_1]_\sigma, \ldots, [t_n]_\sigma\right)$

$w \Vdash_\sigma A \wedge B$ := $w \Vdash_\sigma A$ **and** $w \Vdash_\sigma B$

$w \Vdash_\sigma A \vee B$ := **there exists a bar** $U$ **such that** $w \triangleleft U$ **and for all** $v \in U, v \Vdash_\sigma A$ **or** $v \Vdash_\sigma B$

$w \Vdash_\sigma \top$ := **true**

$w \Vdash_\sigma \bot$ := **there exists a bar** $U$ **such that** $w \triangleleft U$ **and for all** $v \in U$, **false**

$w \Vdash_\sigma A \rightarrow B$ := **for all** $v \geq w$, $\left(v \Vdash_\sigma A\right)$ **implies** $\left(v \Vdash_\sigma B\right)$

$w \Vdash_\sigma \forall x . A$ := **for all** $v \geq w$ **and for all** $a \in M_v, v \Vdash_{\sigma, x \mapsto a} A$

$w \Vdash_\sigma \exists x . A$ := **there exists a bar** $U$ **such that** $w \triangleleft U$ **and for all** $v \in U$ **there exists** $a \in M_v, w \Vdash_{\sigma, x \mapsto a} A$

# Beth Semantics

$w \Vdash_\sigma P(t_1, \ldots, t_n) :=$ **there exists a bar** $U$ **such that** $w \lhd U$ **and for all** $v \in U$, $P_v([t_1]_\sigma, \ldots, [t_n]_\sigma)$

$w \Vdash_\sigma A \wedge B :=$ $w \Vdash_\sigma A$ **and** $w \Vdash_\sigma B$

$w \Vdash_\sigma A \vee B :=$ **there exists a bar** $U$ **such that** $w \lhd U$ **and for all** $v \in U$, $v \Vdash_\sigma A$ **or** $v \Vdash_\sigma B$

$w \Vdash_\sigma \top :=$ **true**

$w \Vdash_\sigma \bot :=$ **there exists a bar** $U$ **such that** $w \lhd U$ **and for all** $v \in U$, **false**

$w \Vdash_\sigma A \to B :=$ **for all** $v \geq w$, $(v \Vdash_\sigma A)$ **implies** $(v \Vdash_\sigma B)$

$w \Vdash_\sigma \forall x . A :=$ **for all** $v \geq w$ **and for all** $a \in M_v$, $v \Vdash_{\sigma, x \mapsto a} A$

$w \Vdash_\sigma \exists x . A :=$ **there exists a bar** $U$ **such that** $w \lhd U$ **and for all** $v \in U$ **there exists** $a \in M_v$, $w \Vdash_{\sigma, x \mapsto a} A$

# Consequences of Beth Semantics

**Monotonicity:**

For any formula $\phi$ and worlds $w \leq v$ in a Beth model, if $w \Vdash_\sigma \phi$ then $v \Vdash_\sigma \phi$

**Local Character:**

For any formula $\phi$ and world $w$ in a Beth model and bar $U \triangleright w$, if $v \Vdash_\sigma \phi$ for all $v \in U$ then $w \Vdash_\sigma \phi$

# Topoi of (Pre)sheaves

Topoi have lots of interesting structure, including an internal higher order logic which can be externalised through Kripke-Joyal semantics. Some results we should thank then:

- Given a category $\mathscr{C}$, its category of presheaves $\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ is a topos
- Given an appropriate notion of covering $\lhd$ in $\mathscr{C}$, its category of sheaves $\mathrm{Sh}\left(\mathscr{C}, \lhd\right)$ is a topos

# Passing to Presheaves

A presheaf $X$ over $\mathbb{W}^{\mathrm{op}}$ consists of the following data

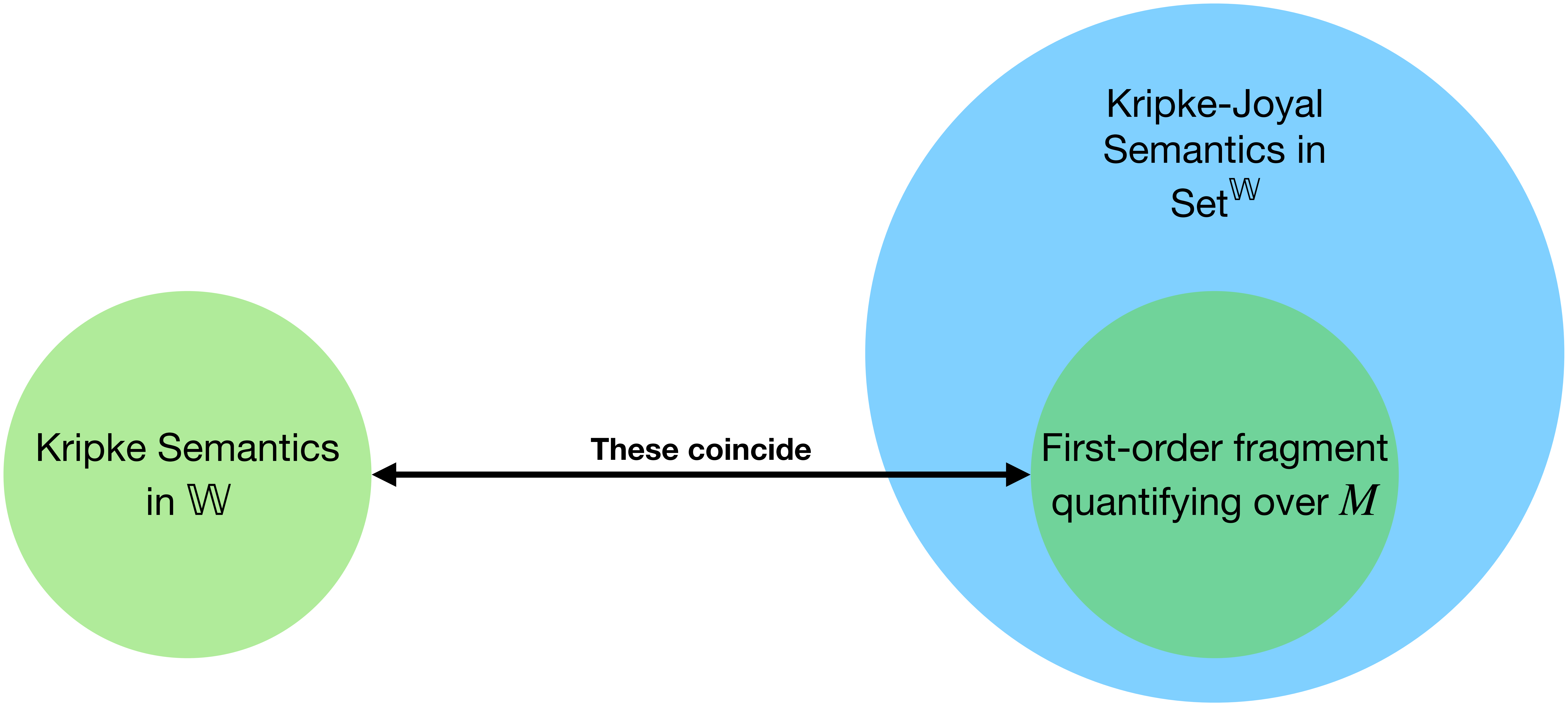
- For each world $w \in \mathbb{W}$, a set $X_w$
- For any worlds $w \leq v$, a transition function $X_{w \leq v} : X_w \to X_v$

Such that the following holds

- For any world $w$, $X_{w \leq w} = \mathrm{id}_{X_w}$
- For any worlds $w \leq v \leq u$, $X_{v \leq u} \circ X_{w \leq v} = X_{w \leq u}$

The domains of discourse $M_w$ with the inclusions $M_w \hookrightarrow M_v$ for $w \leq v$ give such a presheaf.

# Internal Logic of Presheaves

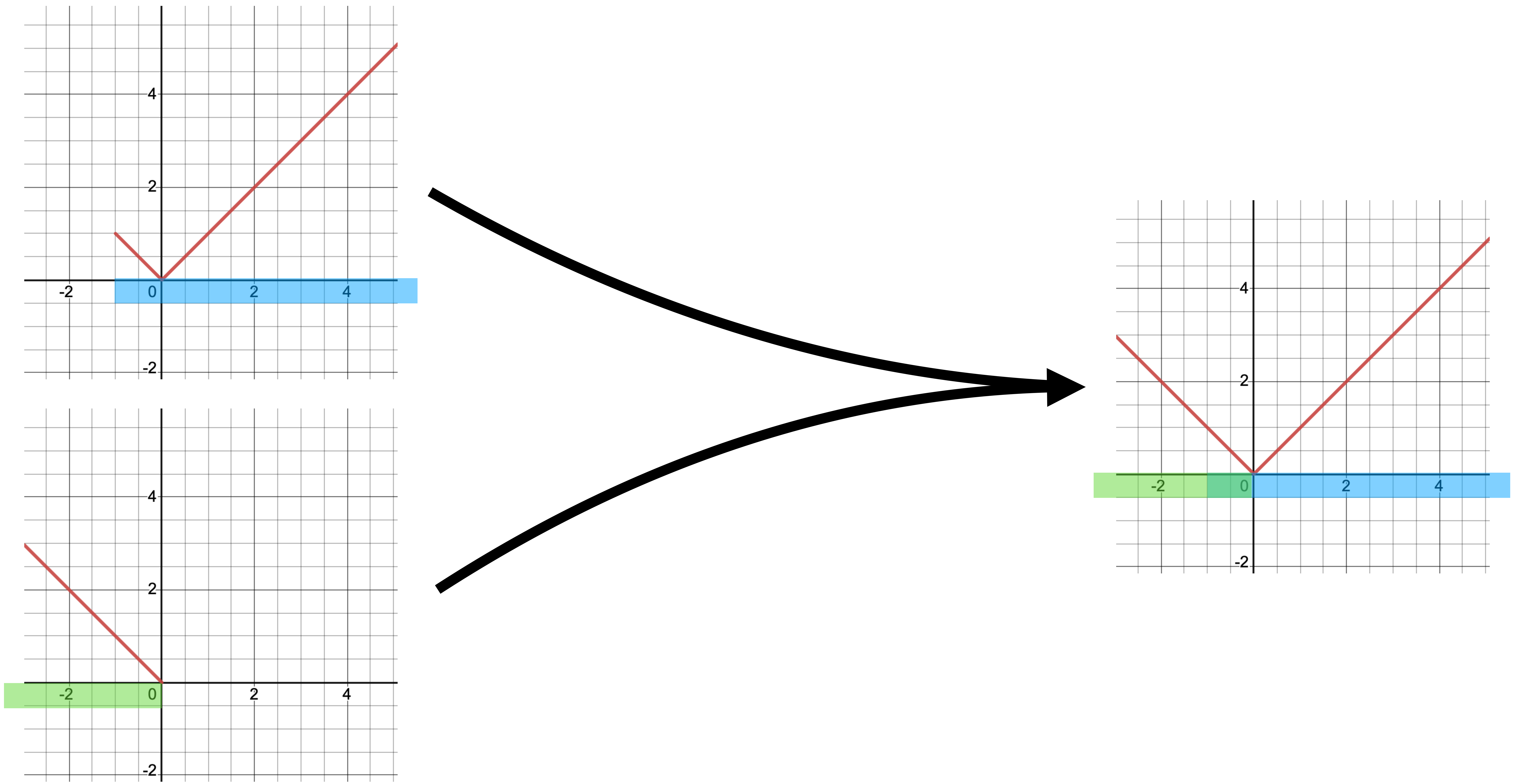


# Passing to Sheaves

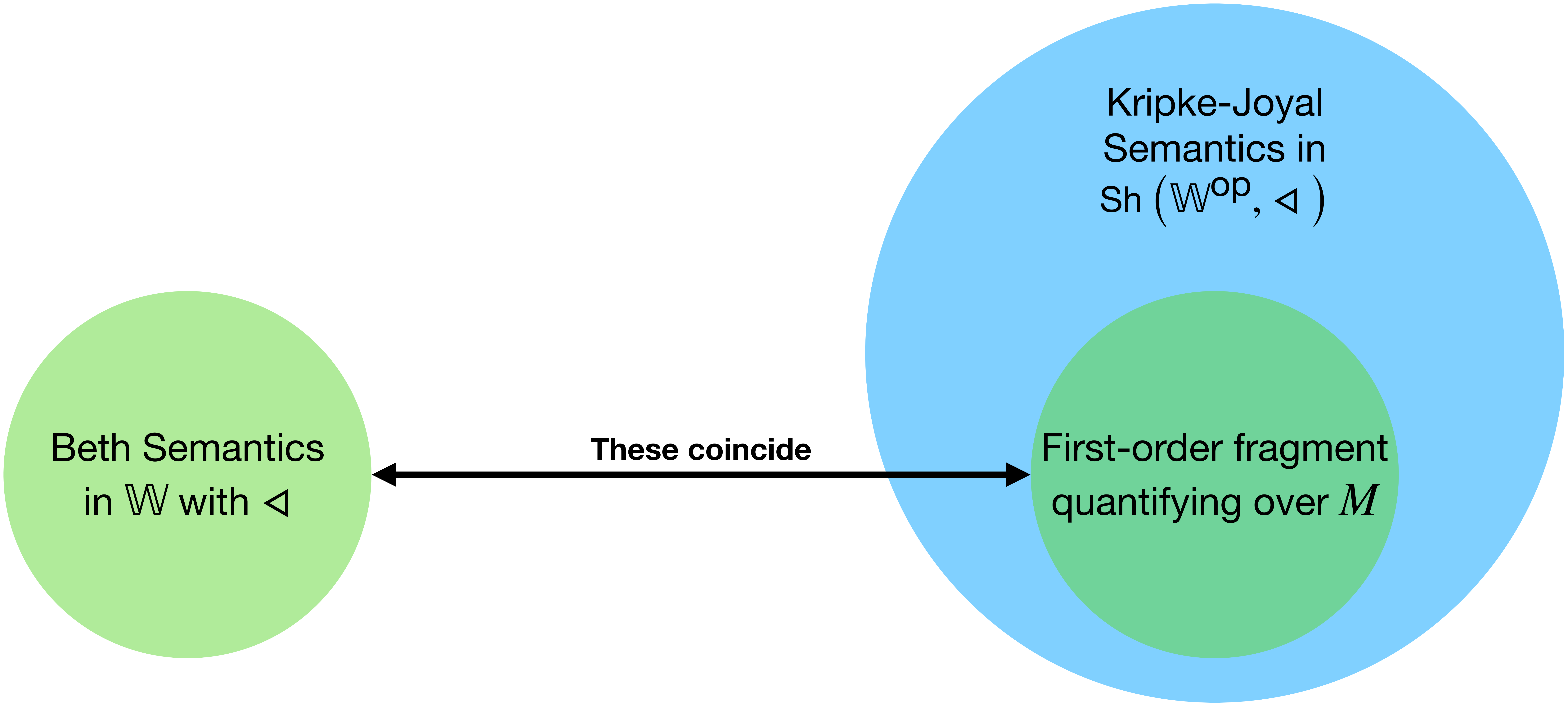
A presheaf $X$ is a sheaf if it satisfies a gluing condition

- Informally, if $w \lhd U$, then given a family of elements $\{x_v \in X_v\}_{v \in U}$ which agree in some technical sense, then they can be glued uniquely to an element $x \in X_w$

# Gluing Condition of Sheaves



# Internal Logic of Presheaves



# Forcing Constructive Principles

- In “A computational interpretation of forcing in Type Theory”, Coquand and Jaber give an intensional type theory validating uniform continuity
  - A “generic” function of type $\mathbb{N} \to \mathbb{N}_2$ is forced whose computation depends on the current world
- In “Realizing Continuity Using Stateful Computations”, Cohen and Rahli give a family of extensional type theories validating restricted forms of continuity and Markov’s principle
  - Functions for reading and updating state are forced

# Forcing Constructive Principles Internally

- The type theory from “A computational interpretation of forcing in Type Theory” can be seen as internal to the category of sheaves on the Cantor space

- The type theories from “Realizing Continuity Using Stateful Computations” can be seen as internal to the category of sheaves on ???

# Bibliography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