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A Model Theoretic Study of Allen's Interval Algebra

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Abstract

Interval algebras were first introduced by Allen to reason about time intervals qualitatively. As a result they have been mainly of interest in computer science, where their applications and questions about decidability have been considered. However, so far they have not been studied in a model theoretic capacity: for example pinning down their axioms and finding their out what properties their models have.

In this report we provide a first order axiomatisation of Allen's interval algebra and we show two constructions: the interval construction $\mathrm{Int}(-)$ sending a linear order to the interval algebra of its non-zero intervals; and the points $\mathrm{Pts}(-)$ construction sending an interval algebra to the linear order of start and end points of its intervals. We show these constructions give rise to a pair of adjoint functors, and we leverage this to show the Fraïssé limit of the finite interval algebras is $\mathrm{Int}(\mathbb{Q})$. From a stability theory perspective, we show that the stable interval algebras are exactly the finite interval algebras and that for any linear order L, $\mathrm{Int}(L)$ has the non-independence property.

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I would also like to thank my second marker Dr. Charlotte Kestner whose feedback on my interim report helped me flesh out the introduction and background sections, turning them into pieces of writing I am very happy with.

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Contents

1 Introduction

1.1 Motivation

Model theory concerns itself with the connection between first order theories, that is sets of axioms possibly using universal and existential quantifiers, and the models of these theories. Some early examples of significant results in model theory can be found in the compactness and Löwenheim–Skolem theorems, all of which focus on the existence of models for a theory: the compactness theorem saying that a model exists for a theory if models exist for each finite fragment of the theory; and the Löwenheim–Skolem theorems saying that if a theory has an infinite model, then there exist models of any cardinality greater than or equal to the number of symbols in the language.

As a consequence of the Löwenheim–Skolem theorems, we know that there exists no theory with a unique infinite model up to isomorphism. This raises the question of what can be said about the number of models a theory has for each cardinality.

Definition 1.1. For a complete theory $\mathbb{T}$, we write $I(\mathbb{T}, \kappa)$ to mean the number of models, up to isomorphism, of $\mathbb{T}$ with cardinality κ. We call $I(\mathbb{T}, -)$ the spectrum of $\mathbb{T}$.

One of the first answers to this question came in the form of Morley's categoricity theorem, which said that for a complete countable theory $\mathbb{T}$, if $I(\mathbb{T}, \kappa) = 1$ for some uncountable κ, then $I(\mathbb{T}, \lambda) = 1$ for all uncountable λ. [1] Work by Shelah in the 1970s then extended this result to uncountable theories [2], beginning the study of classification theory as a discipline of model theory.

Some important notions from stability theory came in the form of stable and unstable theories. Stable theories are well-behaved enough to limit their number of models, while unstable theories always have the maximum number of models, that is if $\mathbb{T}$ is unstable then for any $\kappa > |\mathbb{T}|$ we have $I(\mathbb{T}, \kappa) = 2^{\kappa}$. [3] Most naturally occuring theories in mathematics tend not to be stable though, for example the full theory of any linear order or the theory of real closed fields. Hence it is only natural to study generalisations of stable theories, which will not be as well-behaved but allow us to learn about wider classes of theories. One such generalisation comes from the non-independence property, which, informally, says that there is no formula which can pick out every subset of an infinite subset of a model.

Classification theory is not the only active discipline of model theory however. Another area with a lot of interesting questions comes from the study of homogeneous structures, where any isomorphism between finite substructures can be extended to an automorphism of the homogeneous structure. In 1953, Fraïssé showed how to construct such structures by gluing classes of finite structures together [4], from which their study followed.

Hence, in the spirit of these disciplines, we will consider the first-order theory of interval algebras: a concept first introduced by Allen in 1983 to argue about time qualitatively. We will consider the models of this theory and how well-behaved they are, with the ultimate goal of finding their place in the universe of model-theory.

1.2 Report Structure

The rest of the report will be structured as follows. In Section 2 we summarise the results from Allen's original paper on interval algebras [5], focusing on the choices that lead to the 13 relations and the algorithm to infer missing relations from a given interval network. We also introduce the necessary notions from model theory such as homogeneous models, and stable/NIP theories. We try to keep this introduction grounded by working through some examples where possible. Readers comfortable with these areas may safely skip them in favour of Section 3 and onwards.

In Section 3 we tackle the question of axiomatisation of interval algebras. We propose some axioms based on the understanding from Section 1 and show that these are satisfiable by constructing models from linear orders. To further justify this axiomatisation, we will also show how to extract an underlying linear order from an interval algebra.

In Section 4 we explore further the intervals and points constructions from Section 3 and show they can be extended to two adjoint functors between the relevant categories of models. We give characterisations of the interval algebras and linear orders for which the unit and counit are isomorphisms. These characterisations will be expressible as first order sentences, a fact we will use when studying the classification theory of interval algebras.

Finally, in Section 5 we study the model theory of interval algebras. This begins by considering the the class of finite interval algebras and computing its Fraïssé limit using the machinery developed in Section 4. On the topic of stability, we show that the stable interval algebras are exactly the finite interval algebras. Following this there is a study of the NIP in interval algebras: we find a big class of algebras with the NIP as well as an example of an interval algebra with the IP.

1.3 Ethical Considerations

Model theory, being a subfield of mathematical logic, is an area certainly very theoretical and firmly lodged inside "pure mathematics". As such, the work done in this project has been entirely theoretical and mainly for aesthetic purposes. While Allen's interval algebra is a very concrete construction with a lot of possible real-world applications, none of these applications are expected to benefit from our findings.

And so the main ethical consideration to be had is one familiar to all theoretical projects, that of correctness. Because if our work is both wrong and has no applications, then it will really have been for nothing. Hence a lot of care has been taken to ensure the whole content of this report is correct and that all the arguments hold up to scrutiny. Any omitted proofs have still been checked on paper and so, short of formalising all of this work in a theorem prover, we can be quite sure of its correctness.

2 Background

2.1 Allen's Interval Algebra

The concept of Allen's interval algebras was first introduced in [5]. The main idea was to introduce a new system for arguing about time intervals in a qualitative way, similar to how humans think about time. For example, when retelling a story people will convey some ordering of events' start and end times, while skipping over the actual figures. Now, this might happen because the specific time frames aren't relevant to the story or even because they are not known. Regardless of the reason, it is often helpful to talk about time without being overly explicit. Computers however, tend to argue about time in a quantitative manner: saving timestamps inside of logs, such that if something goes wrong, the order of events can be gotten; checking the system clock to tell if access tokens have expired; so on. The difficulties of expressing time qualitatively on a computer become even more obvious when working towards artificial intelligence, for instance in natural language processing, where the computer must be able to infer the timing of events from people's informal speech or writing.

Interval algebras were Allen's approach to have computers reasoning about time as humans did and still do. In an interval algebra, time intervals are treated as primitives, with binary relations recording their ordering and level of overlap. There are 13 basic relations which allow one to describe fully how any two intervals I and J might relate. The relations and their meaning can be found in Table 1. Apart from $=$ which is its own dual relation, relations come in pairs of dual relations: as an example, if it is known that $I < J$, then it can be immediately infered that $J > I$. Due to this duality we have omitted the meaning of the dual relations from Table 1 to save space. We have also used I_- and I_+ as shorthands for the start and end points of the interval I.

Relation	Symbol	Dual Symbol	Meaning
I starts J	$I < J$	$I > J$	$I_- < I_+ < J_- < J_+$
I meets J	I m J	I M J	$I_- < I_+ = J_- < J_+$
I overlaps J	I o J	I O J	$I_- < J_- < I_+ < J_+$
I starts J	I s J	I S J	$J_- < I_- < I_+ < J_+$
I finishes J	I f J	I F J	$I_- = J_- < I_+ < J_+$
I during J	I d J	I D J	$J_- < I_- < I_+ = J_+$
I equals J	$I = J$	$I = J$	$I_- = J_- < I_+ = J_+$

Table 1: 13 Basic Relations of Allen's Interval Algebra

The choice of these 13 relations has some advantages which simplify reasoning, namely the relations are both exhaustive and mutually-exclusive. By exhaustive we mean that for any two intervals I and J, there exists at least one relation R such that I R J. By mutually exclusive we mean there exists at most one such relation R for any two intervals I and J.

Looking at the intended meaning of each relation in Table 1 it seems that time points are not considered as valid intervals since I_- is always strictly less than I_+. This is no accident, as time points can cause ambiguities and thus complicate the semantics of interval algebras. This deserves some attention, since in natural language we often refer to actual points in time instead of intervals. For example the phrase "I caught the ball" suggests that the act of catching the ball was instantaneous, even though it definitely happened over an interval of time, simply a very short one. In reality, whether we consider certain intervals as time points or as actual intervals depends a lot on context. Furthermore, if one really wishes to deal with time points, then they can be modelled by suitably small intervals.

2.1.1 Basic Algorithm

With some of the main ideas explained, now is a good time to see the basic algorithm described in [5]. It takes as input a collection of intervals and their known relations, and attempts to infer as many of the missing relations as possible. As mentioned before, time intervals are primitives in Allen's interval algebra and since every two intervals must be related by one of the symbols in Table 1, directed graphs with labeled edges are a good representation of this information. In such a graph, there will be a node for each time interval we are interested in, and each edge will be labelled with the sets of symbols that describe the possible relations between the source and target intervals. Considering an illustrative example, suppose there are two nodes I and J with an edge from I to J labelled by "<mo": this should be read as saying that one of $I < J$, $I \text{ m } J$ or $I \text{ o } J$ is expected to hold, although we do not know which one. This labelling takes advantage of the fact that all the basic relations are mutually exclusive, so the implied disjunctions in the above notation cause no ambiguities. Similarly, the fact that the relation symbols are exhaustive also has a consequence for this representation: naïvely, we would expect the interval graph to be a complete directed graph, but as the 13 basic relations all have a dual, it is always possible to tell what the label from I to J should be by reading the label from J to I. As a result, with some care, one can use an undirected graph, allowing for some saved space.

Before seeing the main algorithm it will be helpful to define a helper function: when given two relation symbols $r1$ and $r2$ linking three intervals by $I\ r1\ J$ and $J\ r2\ K$, it is important to tell what constraints there are on the possible relations between I and K. This is done by looking up the relevant entry $T(r1, r2)$ of Table 2. Next, given a pair of edges from I to J and J to K, each labelled by the strings $R1$ and $R2$, we let $\mathit{Constraints}(R1, R2)$ be the minimum set of relation symbols which could relate I and K. The pseudocode for computing this can be found in Algorithm 1.

As for the actual algorithm in question, the pseudocode to update a temporal network with a new label for a specific edge can be seen in Algorithm 2. It is assumed there exists a *ToDo* queue where where we place edges whose constraints need to be updated. Furthermore, for intervals i and j, we let $N(i, j)$ denote the label on the edge from i to j in the interval graph and use $R(i, j)$ to denote the new label for the edge from i to j. Lastly, in simpler cases the *Comparable* function can be taken to always return true, but as the number of intervals grows it can be used for some optimisations discussed in [5, Chapter 5].

	<	m	o	s	f	d	>	M	O	S	F	D
<	<	<	<	<	< m o s d	< m o s d	full	< m o s d	< m o s d	<	<	<
m	<	<	<	m	o s d	o s d	> M O S D	f = F	o s d	m	<	<
o	<	<	< m o	o	o s d	o s d	> M O S D	O S D	con	o F D	< m o	< m o F D
s	<	<	< m o	s	d	d	>	M	f d O	s = S	< m o	< m o F D
f	<	m	o s d	d	f	d	>	>	> M O	> M O	f = F	> M O S D
d	<	<	< m o s d	d	d	d	>	>	d f > O M	d f > O M	< m o s d	full
>	full	d f > O M	d f > O M	d f > O M	>	d f > O M	>	>	>	>	>	>
M	< m o F D	s = S	f d O	f d O	M	f d O	>	>	>	>	M	>
O	< m o F D	o F D	con	f d O	O	f d O	>	>	> M O	> M O	O S D	> M O S D
S	< m o F D	o F D	o F D	s = S	O	f d O	>	M	O	S	D	D
F	<	m	o	o	f = F	o s d	> M O S D	O S D	O S D	D	F	D
D	< m o F D	o F D	o F D	o F D	O S D	con	> M O S D	O S D	O S D	D	D	D

Table 2: Transitivity table for basic relations – adapted from [6].
Given relations $I\ r1\ J$ and $J\ r2\ K$, the possible relations between I and K will be under row $r1$ and column $r2$. Entries with the word "full" should contain all relations and entries labelled "con" should contain "o s f d = O S F D".

Algorithm 1 Computing constraints given two edge labels. [5]

```
function CONSTRAINTS(R1, R2)
    C ← ∅
    for all r1 ∈ R1 do
        for all r2 ∈ R2 do
            C ← C ∪ T(r1, r2)
        end for
    end for
    return C
end function
```

Algorithm 2 Updating temporal network. [5]

```
procedure TO ADD(R(i, j))
    Add (i, j) to queue ToDo
    while ToDo is not empty do
        Get next (i, j) from queue ToDo
        N(i, j) ← R(i, j)
        for all nodes k such that COMPARABLE(k, j) do
            R(k, j) ← N(k, j) ∩ CONSTRAINTS(N(k, j), R(i, j))
            if R(k, j) ⊂ N(k, j) then
                Add (k, j) to ToDo
            end if
        end for
        for all nodes k such that COMPARABLE(i, k) do
            R(i, k) ← N(i, k) ∩ CONSTRAINTS(R(i, j), N(j, k))
            if R(i, k) ⊂ N(i, k) then
                Add (i, k) to ToDo
            end if
        end for
    end while
end procedure
```

With an inference algorithm such as this, the question of correctness is quite important. Thankfully, if given a satisfiable network of intervals, this algorithm will never infer any erroneous constraints. However, if given an unsatisfiable network, then it is not guaranteed that the algorithm will detect the unsatisfiability. Since we only consider paths of length 2, any inconsistencies detected must occur within a 3 node subgraph and inconsistencies from larger subgraphs will be missed. If one is unsure about the satisfiability of the input graph though, then a search for valid assignments can be made after running the inference algorithm. It will still be worth it to run the inference algorithm in this case since the extra added constraints might make it a lot faster to check for valid assignments.

2.2 Model Theory

We assume some familiarity with the basics of model theory, for which we recommend [7]. In this section we will focus mainly on the definitions which might not feature in an introductory course in model theory, but are needed to understand the work done in Section 5.

First we introduce the idea of homogeneous structures, and how to construct examples of these using Fraïssé limits. These structures will be characterised by their ability to extend isomorphisms between finite substructures to automorphisms of the whole structure. As a result, homogeneous structures can have very interesting automorphism groups and their study helps link model theory, group theory and combinatorics. For a thorough survey of the area see [8].

Then we will cover some important classes of theories arising from classification theory, namely we will introduce stable theories and one of their possible generalisations, NIP theories. The concept of stable theories arose to study the spectrum of complete theories, offering a definitive answer through the use of tools like forking and dividing. NIP theories generalise the class of stable theories to include important examples like linear orders and geometric examples like algebraically closed valued fields [9] or the real exponential field [10].

2.2.1 Homogenous Structures and Fraïssé Classes

We work with the definition of a homogenous structure found in [8] as we are mainly interested in the study of Fraïssé limits.

Definition 2.1. A homogenous structure is a countable, possibly finite, relational structure (with finite language $\mathcal{L}$) such that, for every isomorphism $f : U \to V$ between finite substructures $U, V \subseteq M$, there is an authomorphism $f' : M \to M$ on M extending f.

The simplest example of a homogeneous structure comes from the theory of strict linear orders, which we now define.

Definition 2.2. We define the language of strict linear orders $\mathcal{L}_{\text{SLO}}$ as the single binary relation $\{<\}$.

Definition 2.3. We define the theory of strict linear orders as

$$\begin{aligned}\mathbb{T}_{\text{SLO}} = \{\ & \forall a, \neg\ (a < a),\\ & \forall a, \forall b, \forall c, (a < b) \wedge (b < c) \to (a < c)\\ & \forall a, \forall b, (a < b) \vee (a = b) \vee (b < a)\ \}\end{aligned}$$

So a strict linear order consists of a binary relation on a set, which is irreflexive, transitive satisfies the trichotomy condition. Notice that from these we can infer that the binary relation must be antisymmetric, since if $a < b$ and $b < a$ then by transitivity $a < a$, which contradicts the irreflexivity of $<$.

Proposition 2.4. *The strict linear order* $(\mathbb{Q}, <)$ *is homogeneous.*

Proof. First we see how to expand the domain of an order isomorphism $f : L \to P$ with $L, P \subseteq \mathbb{Q}$ both finite. Suppose that we wish to extend the domain of f to include some $a \in \mathbb{Q} \setminus L$. There are three cases to worry about here:

- If a is an upper bound for L, we fix some upper bound b of P which is not in P. Such a b must exist as $\mathbb{Q}$ is unbounded and P is finite. Then, we extend $f : L \to P$ to $g : L \cup \{a\} \to P \cup \{b\}$ by sending $g(a) = b$. This remains an order isomorphism since for any $x \in P$, we must have $x < a$ and $g(x) < b = g(a)$ since a and b are upper bounds of L and P respectively.
- If a is a lower bound for L, then we fix a lower bound b of P not already in P and extend f to $g : L \cup \{a\} \to P \cup \{b\}$ by sending a to b again. Dually to the upper bound case, a and b are lower bounds of the domain and codomain, hence g remains an order isomorphism.
- If a is neither an upper nor lower bound, then we notice that L, P are finite linear orders and hence discrete. This means that we can find $a_1, a_2 \in L$ such that $a_1 < a < a_2$ and for no $x \in L$ do we have $a_1 < x < a_2$. Now we can fix some $b \in \mathbb{Q} \setminus P$ such that $g(a_1) < b < g(a_2)$ since $\mathbb{Q} \setminus P$ is still dense. We extend f to $g : L \cup \{a\} \to P \cup \{b\}$ by sending g(a) = b. This remains an order isomorphism since for any $x \in L$ we have either $x \leq a_1 < a$, so

  $$g(x) = f(x) \leq f(a_1) = g(a_1) < b = g(a)$$

  or we have $a < a_2 \leq x$, in which case

  $$g(a) = b < g(a_2) = f(a_2) \leq f(x) = g(x)$$

If we wanted to expand the codomain of f then it suffices to extend the domain of f^{-1} to include whichever element we needed, giving us a function g. The inverse g^{-1} is then the required extension of f.

Now, fix two finite suborders $L, P \subseteq \mathbb{Q}$ and suppose we have some order isomorphism $f : L \to P$. To extend f to an automorphism of $\mathbb{Q}$ we start by fixing an enumeration $(a_1, a_2, \dots)$ of the elements of $\mathbb{Q}$ and we define three sequences: two sequences $(L_1, L_2, \dots)$ and $(P_1, P_2, \dots)$ of increasing subsets of $\mathbb{Q}$; and a sequence $(g_1, g_2, \dots)$ of bijections $g_i : L_i \to P_i$ where each g_i extends its predecessors. These sequences are defined inductively by:

- $k = 1$: let $L_1 = L$, $P_1 = P$ and $g_1 = f$.
- $k = 2l$ for $l \in \{1, 2, \dots\}$: at even indices we focus on increasing the domain of g_i to

all of $\mathbb{Q}$. If $a_l \in L_{k-1}$ already then we let

$$\begin{aligned} L_k &= L_{k-1} \\ P_k &= P_{k-1} \\ g_k &= g_{k-1} \end{aligned}$$

Otherwise we extend g_{k-1} to an order isomorphism $h : L_{k-1} \cup \{a_l\} \to P_{k-1} \cup \{b\}$ for some appropriate b and let

$$\begin{aligned} L_k &= L_{k-1} \cup \{a_l\} \\ P_k &= P_{k-1} \cup \{b\} \\ g_k &= h \end{aligned}$$

- $k = 2l + 1$ for $l \in \{1, 2, \dots\}$: at odd indices we focus on increasing the codomain of g_i to all of $\mathbb{Q}$. If $a_l \in P_{k-1}$ already then we let

  $$\begin{aligned} L_k &= L_{k-1} \\ P_k &= P_{k-1} \\ g_k &= g_{k-1} \end{aligned}$$

  Otherwise we extend g_{k-1} to an order isomorphism $h : L_{k-1} \cup \{b\} \to P_{k-1} \cup \{a_l\}$ for some appropriate b and let

  $$\begin{aligned} L_k &= L_{k-1} \cup \{b\} \\ P_k &= P_{k-1} \cup \{a_l\} \\ g_k &= h \end{aligned}$$

We must have $\bigcup L_k = \bigcup P_k = \mathbb{Q}$ since each $x \in \mathbb{Q}$ must appear as a_l in our enumeration for some $l \in \{1, 2, \dots\}$. So $x \in L_{2l}$ and $x \in P_{2l+1}$ implying that $x \in \bigcup L_k$ and $x \in \bigcup P_k$. Since each g_k is an isomorphism, their union $g = \bigcup g_k$ must also be and by construction g will extend f. $\square$

The main bulk of the work above comes from the density of $\mathbb{Q}$, in fact, non dense linear orders are only homogeneous in the trivial case.

Proposition 2.5. *If I is a non-dense strict linear order with more than one element, then L is not homogeneous.*

Proof. If the linear order is $\{x < y\}$ then the map partial isomorphism $x \mapsto y$ cannot be extended further.

If the linear order has more than 3 elements, we can either find $x < y < z$ or $z < x < y$ where x and y have no elements inbetween (as the order is not dense). In the first case, the partial order isomorphism

$$f : \{x, z\} \to \{x, y\} \text{ sending } x \mapsto x \text{ and } z \mapsto y.$$

cannot be extended to add y since there is no w such that $f(x) = x < w < y = f(z)$. By a similar argument, the following map handles the second case

$$f : \{y, z\} \to \{x, y\} \text{ sending } y \mapsto y \text{ and } z \mapsto x.$$

□

Corollary 2.6. $\mathbb{N}$ *and* $\mathbb{Z}$ *are not homogeneous.*

Homogeneous structures can seem quite mysterious at first, so it would be interesting to see how to build homogeneous models of a theory. Towards this goal, fix some relational language $\mathcal{L}$ and structure M and consider the class of finite $\mathcal{L}$-structures embeddable intro M. This will be called the age of the structure M, denoted $\mathrm{Age}(M)$. In general, the age of any countable relational structure will satisfy the following three properties.

Definition 2.7. A class $\mathcal{C}$ has the hereditary property (HP) if it is closed under substructures, so if A, B are $\mathcal{L}$-structures, $A \in \mathcal{C}$ and $B \subseteq A$ then $B \in \mathcal{C}$ too.

Definition 2.8. A class $\mathcal{C}$ has the joint embedding property (JEP) if for any $A, B \in \mathcal{C}$, we can find a third $\mathcal{L}$-structure $C \in \mathcal{C}$ such that A and B both embed into it. language $\mathcal{L}$).

Definition 2.9. A class $\mathcal{C}$ is essentially countable (EC) if, up to isomorphism, there are only countably many $\mathcal{L}$-structures.

Given a class $\mathcal{C}$ satisfying the above three properties, one may construct a countable model M such that $\mathrm{Age}(M) = \mathcal{C}$ by enumerating all the isomorphism classes in $\mathcal{C}$ and gluing all of these in order using the JEP. Suppose that M is homogeneous though, then we can say something further about its age, namely it will satisfy the following.

Definition 2.10. A class $\mathcal{C}$ has the amalgamation property (AP) if for any span

$$A \longleftarrow C \longrightarrow B$$

with $A, B, C \in \mathcal{C}$, there exists some $\mathcal{L}$-structure $\Omega \in C$ with embeddings

$$A \longrightarrow \Omega \longleftarrow B$$

making the following diagram commute:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & \Omega \end{tikzcd}$$

Considering again a class of models $\mathcal{C}$ satisfying the HP, EC, and JEP, suppose further that that $\mathcal{C}$ also has the AP. Then a theorem of Fraïssé tells us that gluing all of the models in $\mathcal{C}$ yields a homogeneous model M with $\mathrm{Age}(M) = \mathcal{C}$.

Theorem 2.11 (Fraïssé's theorem)**.** *Given a class $\mathcal{C}$ of finite $\mathcal{L}$-structures satisfying the properties HP, JEP, AP and EC, then there exists some homogeneous $\mathcal{L}$-structure M such that* $\mathrm{Age}(M) = \mathcal{C}$. *Furthermore, if we have two such homogeneous $\mathcal{L}$-structures M and N, then necessarily $M \cong N$.*

A proof of this, along with proofs that $\mathrm{Age}(M)$ satisfies the relevant properties can be found in [11]. When the above structure M exists for a class $\mathcal{C}$, then we call M the Fraïssé limit of $\mathcal{C}$.

Note that the uniqueness condition of the Fraïssé limit only applies to homogeneous structures. In fact, if we consider the class of finite strict linear orders, we can see it coincides with $\mathrm{Age}(\mathbb{Q})$, $\mathrm{Age}(\mathbb{Z})$ and also $\mathrm{Age}(\mathbb{N})$ despite neither of these linear orders being isomorphic. We saw earlier that neither $\mathbb{Z}$ nor $\mathbb{N}$ were homogeneous though, hence why this happens.

As we have decided not to show a lot of proofs for this section, we will instead consider in detail how to compute the Fraïssé limit of the class **FCh** of finite strict linear orders, to hopefully ground all these definitions a bit better. First we need to see this limit exists by checking that **FCh** satisfies the 4 properties needed to apply Fraïssé's theorem.

First we prove some general results about the HP and EC:

Proposition 2.12. *Given a relational, universal theory $\mathbb{T}$, the class of finite models of $\mathbb{T}$ satisfies the hereditary property.*

Proof. Fix some finite model $M \models \mathbb{T}$ and a substructure $N \subseteq M$. Since M was finite, N must also be finite. Now, suppose we have a universal sentence

$$\phi = \forall x_1, \ldots, \forall x_n, \psi(x_1, \ldots, x_n)$$

where ψ is a quantifier free formula such that $M \models \phi$. Fixing some tuple $(a_1, \ldots, a_n) \in N^n$ we know that $M \models \psi(a_1, \ldots, a_n)$, but N is a substructure of M and the truth value of quantifier free formulas is preserved by embeddings, so $N \models \psi(a_1, \ldots, a_n)$ too. As we fixed an arbitrary tuple of elements in N, this means that $N \models \phi$. The theory $\mathbb{T}$ is taken to be universal, hence all sentences $\phi \in \mathbb{T}$ are equivalent to some universal sentence modulo $\mathbb{T}$. As we just saw, universal sentences are preserved under taking substructures, so ϕ must be preserved too. Hence if $M \models \mathbb{T}$, then any substructure $N \subseteq M$ will also be a model of $\mathbb{T}$. □

Corollary 2.13. *The class **FCh** has the HP.*

Proposition 2.14. *For any finite language $\mathcal{L}$, the class of all finite $\mathcal{L}$-structures is essentially countable.*

Proof. Fix some finite language

$$\mathcal{L} = \{c_1, \ldots, c_m, f_1, \ldots, f_n, R_1, \ldots, R_l\}$$

where the c_i are constant symbols, the f_i are function symbols and the R_i are relation symbols. We will show that the class of finite $\mathcal{L}$-structures must be essentially countable. First, consider the initial segment $M = \{1, \ldots, r\} \subset \mathbb{N}$. If we wish to turn M into a $\mathcal{L}$-structure then we must pick interpretations for all the symbols in $\mathcal{L}$. For a constant symbol this consists of picking a single element, so there are r possible options. For a function symbol with a-arity, we need to pick an element of M for every input tuple of a elements of M, so there are r^{r^a} possibilities. Similarly, for a relation symbol R with b-arity, we need to pick a truth value for all tuples of size b in M, so there are 2^{r^b} options. Hence, if we denote by a_i the arity of the symbol f_i and b_i the arity of the symbol R_i, then the total number of distinct $\mathcal{L}$-structures on M is finite, more specifically it is $r^{\left(m+\sum_{i=1}^{n} r^{a_i}\right)} 2^{\left(\sum_{i=1}^{l} r^{b_i}\right)}$. As such, the set of possible $\mathcal{L}$-structures on all initial segments $\{1, \ldots, r\} \subseteq \mathbb{N}$ is countable. Every finite $\mathcal{L}$-structure must be isomorphic to at least one of these, so the class of finite $\mathcal{L}$-structures is essentially countable too. □

Corollary 2.15. *The class **FCh** is EC.*

Proof. The class **FCh** is a subclass of all finite strict linear orders, hence if the latter is EC, then the former must also be EC. □

Now we focus solely on strict linear orders.

Proposition 2.16. *The class **FCh** has the joint embedding property.*

Proof. Given two strict linear orders L, P, we can turn their disjoint union $L \sqcup P$ into a strict linear order by carrying over the orderings from L and P and setting $x < y$ for all $x \in L$ and $y \in P$. Then, the usual injections into the disjoint union $f : L \to L \sqcup P$ and $g : P \to L \sqcup P$ are order preserving maps. If L and P are both finite, so will $L \sqcup P$ be, hence the class of finite strict linear orders has the joint embedding property. □

Proposition 2.17. *The class **FCh** has the amalgamation property.*

Proof. Given linear orders A, L, P and order embeddings $f : A \to L$, $g : A \to P$, we wish to find a linear order Ω and order embeddings $f' : L \to \Omega$, $g' : P \to \Omega$ such that $f' \circ f = g' \circ g$. If any of A, L, P is empty, we devolve to the previous proof, so we may assume that A, L and P are all non-empty. We take the product $L \times P$ and order it lexicographically, such that for all $(x, y), (x', y') \in L \times P$

$$(x, y) < (x', y') \iff (x < y') \vee ((x = x') \wedge (y < y'))$$

Fixing some $y \in P$ we define $f' : L \to L \times P$ by

$$f'(x) = \begin{cases} (x, g(a)) & \text{if} f^{-1}(x) = \{a\} \\ (x, y) & \text{if } f^{-1}(x) = \emptyset \end{cases}$$

This preserves the ordering of L as it coincides with the identity map on the first coordinate and we are ordering $L \times P$ lexicographically.

Defining $g' : P \to L \times P$ is slightly trickier. First, we must pick some $y_{L,U} \in L$ such that $L \leq y_{L,U} \leq U$ for all subsets $L, U \subseteq L$ with $L < U$. Once we have fixed our choices, for any $x \in P$ we denote $y_x = y_{\{f(a) \ | \ g(a) < x\}, \{f(a) \ | \ x < g(a)\}}$. Finally, we define

$$g'(x) = \begin{cases} (f(a), x) & \text{if} g^{-1}(x) = \{a\} \\ (y_x, x) & \text{if } f^{-1}(x) = \emptyset \end{cases}$$

To check this preserves the ordering of P, we fix two elements $x < x'$ of P, then:

- If $g^{-1}(x) = \{a\}$ and $g^{-1}(x') = \{b\}$, then $a < b$, which implies that $f(a) < f(b)$, so $g'(x) = (f(a), x) < (f(b), x') = g'(x')$.
- If $g^{-1}(x) = \{a\}$ and $g^{-1}(x') = \emptyset$, then we have picked y_x such that $f(a) < y_x$ so $g'(x) = (f(a), x) < (y_x, x) = g'(x')$.
- If $g^{-1}(x) = \emptyset$ and $g^{-1}(x') = \{a\}$, then we have picked y_x such that $y_x < f(a)$ so $g'(x) = (y_x, x) < (f(a), x) = g'(x')$.
- If $g^{-1}(x) = g^{-1}(x') = \emptyset$ then notice that $y_x \leq y_{x'}$ since either there exists some $a \in A$ such that $x < g(a) < x'$, so then $y_x \leq g(a) \leq y_{x'}$ or no such a exists and $y_x = y_{x'}$. Thus, $g'(x) = (y_x, x) < (y_{x'}, x') = g'(x')$.

Now that we have the required order embeddings into $\Omega = L \times P$, we just check the necessary commutativity condition:

$$f' \circ f(a) = f'(f(a)) = (f(a), g(a)) = g'(g(a)) = g' \circ g(a)$$

In the case that A, L, P are finite, then $L \times P$ will also be finite, so this shows that the class of finite strict linear orders has the amalgamation property. □

Theorem 2.18. *The Fraïssé limit of **FCh** is $\mathbb{Q}$ with its usual order.*

Proof. As **FCh** satisfies the HP, JEP, AP and is EC, then it must have a Fraïssé limit M. We saw before that $\mathbb{Q}$ is homogeneous. By uniqueness of the Fraïssé limit, it suffices to show that $\mathrm{Age}(\mathbb{Q}) = \mathbf{FCh}$. Clearly $\mathrm{Age}(\mathbb{Q}) \subseteq \mathbf{FCh}$ since a suborder of a linear order must still be linear. To see that $\mathbf{FCh} \subseteq \mathrm{Age}(\mathbb{Q})$ we fix some finite linear order L, then there is an order preserving isomorphism from L to an initial segment of $\mathbb{N}$. The inclusion $\mathbb{N} \in \mathbb{Q}$ means this isomorphism realises L as a finite suborder of $\mathbb{Q}$. □

2.2.2 Stable Theories

Definition 2.19. Let $\mathbb{T}$ be a complete theory. For an infinite cardinal κ, $\mathbb{T}$ is κ-stable if for every every model M of $\mathbb{T}$ and subset $A \subseteq M$ with cardinality $|A| = \kappa$, then the set of complete n-types in M over A, $S_n^M(A)$, has cardinality κ.

If a theory $\mathbb{T}$ is κ-stable for any infinite cardinal κ, then we call it stable, otherwise it is called unstable. Given a model M, we say that M is stable (resp. unstable) if the full theory of M is stable (resp. unstable).

The following theorem gives us a characterisation of stability in terms of linear orders, which can be simpler to reason with, especially when one considers the close relation between linear orders and interval algebras.

Theorem 2.20. *Let $\mathbb{T}$ be a complete theory, then $\mathbb{T}$ is stable if and only if there exists a formula $\phi(v_1, \dots, v_n; w_1, \dots, w_n)$ and a model $M \models \mathbb{T}$ with a sequence $x_1, x_2, \dots \in M^n$ such that*

$$M \models \phi(x_i; x_j) \iff i < j$$

Such a formula is said to have the order property.

The proof for this is somewhat involved, so we refer the interested reader to [7].

Corollary 2.21. *A linear order L is stable if and only if it is finite.*

Example 2.22. A complete theory is strongly minimal if for all models M, any definable set (with parameters) $D \subseteq M$ is either finite or cofinite. This turns out to be a very strong requirement, meaning that strongly minimal theories must also be stable. The proof for this relies on the equivalence between stability of a theory and non-existence of a formula with the strict order property (which is slightly different from the order property), and so is omitted.

We know from [7] that the following are strongly minimal and hence stable:

- The theory of algebraically closed fields in characteristic p in the language of rings: all definable sets of an algebraically closed field k are boolean combinations of zero sets of polynomials in $k[x]$. Since these zero sets are either finite or all of k, then the claim of strong minimality follows.
- The theory of $\mathbb{Q}$-vector spaces in the language of modules: for a $\mathbb{Q}$-vector space V, all definable sets $D \subseteq V$ are given by boolean combinations of formulas of the form $nx = a$ where $n \in \mathbb{N}$, and $a \in V$. If a is nonzero, such a formula can have at most one solution, hence D will have to be either finite or cofinite.

2.2.3 NIP Theories

Definition 2.23. For a complete theory $\mathbb{T}$, we say that a formula $\phi(x; y)$ has the independence property if there is a model $M \models \mathbb{T}$ and sequences $(a_i)_{i<\omega}$, $(b_I)_{I\subseteq\omega}$ in M such that

$$M \models \phi(a_i, b_I) \iff i \in I$$

If a formula does not have the IP, we say it has the non-independence property (NIP).

Definition 2.24. A complete theory has the IP if there exists some formula with the IP. It has the NIP if all formulas have the NIP.

By compactness, to show that a specific formula has the IP, it suffices to consider arbitrarily large finite sequences.

Proposition 2.25. *For a complete theory $\mathbb{T}$, a formula $\phi(x; y)$ has the IP if and only if the following is satisfiable*

$$\mathbb{T} \cup \{\phi(a_i, b_I) \mid i \in I \subseteq \{0, 1, \ldots, n\}\}$$

for arbitrarily large n (where the a_i and b_I are new constant symbols).

Proof. The forwards implication follows by taking the model and sequences $(a_i)_{i<\omega}$, $(b_I)_{I\subseteq\omega}$ which realise the IP for $\phi(x; y)$ and discarding the a_i with $i > n$ and $I \nsubseteq \{0, 1, \ldots, n\}$.

For the converse, showing that $\phi(x; y)$ has the IP amounts to showing the satisfiability of

$$\mathbb{T} \cup \{\phi(a_i, b_I) \mid i \in I \subseteq \mathbb{N}\}$$

But by compactness, it suffices to show satisfiability of

$$\mathbb{T} \cup \{\phi(a_{i_1}, b_{I_1}), \phi(a_{i_2}, b_{I_2}) \ldots, \phi(a_{i_m}, b_{I_m})\}$$

where $i_k \in I_k$ for all k. Letting $n = \max(i_1, i_2, \ldots, i_m)$ and applying our hypothesis, we see this is indeed satisfiable. □

This means that if a theory has the NIP, then for all formulas $\phi(x; y)$ and all models M, there exists a maximum $n \in \mathbb{N}$ such that for all $\{a_1, \ldots, a_n\} \subseteq M$, there exists some subset of $\{a_1, \ldots, a_n\}$ which we cannot pick out with $\phi(x; y)$ regardless of the parameter y.

Being a generalisation of stable theories, we expect all stable theories to also have the NIP.

Proposition 2.26. *All stable theories have the NIP.*

Proof. Suppose we have a theory $\mathbb{T}$ which has the IP, so the IP is realised for some formula $\phi(x; y)$ by a model $M \models \mathbb{T}$ and elements $(a_i)_{i<\omega}$, $(b_I)_{I\subseteq\omega}$. Then the formula

$$\psi(x, x'; y, y') = \phi(x; y')$$

is unstable, since if the sequence $(c_i)_{i<\omega}$ given by $c_i = (a_i, b_{\{n|n<i\}})$ is such that

$$M \models \psi(c_i; c_j) \iff M \models \phi(a_i; b_{\{n|n<j\}}) \iff i \in \{n \mid n < j\} \iff i < j$$

This shows that ψ has the order property, so $\mathbb{T}$ must be unstable. □

However not all NIP theories are stable, for example all linear orders, infinite or not, are NIP. The proof of this requires some machinery which is not relevant to the rest of this project, but the details can be found in [12].

Example 2.27. The real exponential field $(\mathbb{R}, +, \cdot, 0, 1, e^x)$ has the NIP. [10] This means that for any formula $\phi(x; y)$ there exists some maximum n such that

$$\mathrm{Th}(\mathbb{R}) \cup \{\phi(a_i, b_I) \mid i \in I \subseteq \{0, 1, \ldots, n\}\}$$

is satisfiable. We refer to this n as the VC dimension of $\phi(x; y)$. This notion of VC dimension is not only relevant in model theory, in machine learning it can be used to measure the expressive power of a classification model. One interesting and often studied class of classification models comes from feedforward neural networks with sigmoid activation functions. It turns out that any such neural network can be expressed as a first order formula in the language of exponential fields with its weights as parameters [13] and as such any feedforward neural network with sigmoid activation functions will have finite VC dimension.

Interestingly, by modifying the above slightly to consider the complex exponential field, we get a theory with the IP. This points towards the "precariousness" of the NIP, and how even small changes in the theory may result in losing this property.

Example 2.28. To see why the complex exponential field $(\mathbb{C}, +, \cdot, 0, 1, e^x)$ has the IP, we first see how to define the set of integers. For this, notice that the set $\{i, -i\}$ is defined by the formula $\phi(x) = x \cdot x = 1$. Then, recall that we have

$$\sin(\theta) = \frac{e^{i\theta} - e^{(-i)\theta}}{2i} = \frac{e^{(-i)\theta} - e^{i\theta}}{2(-i)}$$

hence we can define sine using the formula

$$\psi(x, y) = \exists a, \exists b, (a \neq b) \wedge \phi(a) \wedge \phi(b) \wedge \left(y = \frac{e^{ax} - e^{bx}}{2a} \right)$$

In turn, this allows us to define $2\pi\mathbb{Z} \subseteq \mathcal{C}$ as the zero set of the sine function. Finally, the following formula, which essentially says that as integers, y divides x, has the IP

$$\varphi(x; y, \pi) = \exists z, \psi(z, 0) \wedge \left(y \cdot \frac{z}{2\pi} = x \right)$$

which we show by considering arbitrarily large finite sets. Fix some n, then the subsets of $\{1, \ldots, n\}$ can be put in bijection with $\{1, \ldots, n^2\}$ under some f. Using p_k to refer to the kth prime, we let

$$b_I = (p_{f(I)}, \pi)$$

for each $I \subseteq \{1, \ldots, n\}$. Also for each $i \in \{1, \ldots, n\}$ we let

$$a_i = \prod_{\substack{J \subseteq \{1, \ldots, n\}, \\ i \in J}} p_{f(J)}$$

With the use of these parameters

$$\mathcal{C} \models \varphi(a_i; b_I) \iff p_{f(I)} \mid \prod_{\substack{J \subseteq \{1,\ldots,n\},\\ i \in J}} p_{f(J)} \iff i \in I$$

As this works for arbitrarily n, $\varphi(x; y, a)$ must have the IP.

3 Axiomatisation of Interval Algebras

The first step we must take is to define the language of interval algebras. Since equality is always assumed to be available in first order logic, we only need symbols for the 12 remaining binary relations.

Definition 3.1. We define the language of Allen interval algebras $\mathcal{L}_{\text{AIA}}$ as

$$\mathcal{L}_{\text{AIA}} = \{\xrightarrow{<}, \xrightarrow{m}, \xrightarrow{o}, \xrightarrow{s}, \xrightarrow{f}, \xrightarrow{d}, \xleftarrow{<}, \xleftarrow{m}, \xleftarrow{o}, \xleftarrow{s}, \xleftarrow{f}, \xleftarrow{d}\}$$

The theory of Allen interval algebras follows closely with Allen's main concerns in his original paper. We must specify that our relation symbols are both exhaustive and mutually exclusive. It is also important that our dual relation symbols act as such, otherwise the reasoning possible would not match with the reasoning done by the interval algebra algorithm. Finally, we need to encode the many transitivity-like requirements, which is another crucial feature of the algorithm we saw.

Letting $I = \{<, m, o, s, f, d, =, >, M, O, S, F, D\}$ and using $\xrightarrow{=}$ to mean $=$, we can express notion of exhaustability with the sentence

$$\phi_{\text{exh}} = \forall I,\ \forall J,\ \bigvee_{i \in I} (I \xrightarrow{i} J)$$

The mutual exclusivity requirement is expressed by

$$\phi_{\text{mutex}} = \forall I,\ \forall J,\ \bigwedge_{\substack{i, j \in I \\ i \neq j}} \neg \Big((I \xrightarrow{i} J) \wedge (I \xrightarrow{j} J) \Big)$$

The duality is given by the following sentences

$$\begin{aligned}
\phi_{\text{dual}_>} &= \forall I,\ \forall J,\ (I \xrightarrow{<} J) \leftrightarrow (J \xleftarrow{<} I) \\
\phi_{\text{dual}_m} &= \forall I,\ \forall J,\ (I \xrightarrow{m} J) \leftrightarrow (J \xleftarrow{m} I) \\
\phi_{\text{dual}_o} &= \forall I,\ \forall J,\ (I \xrightarrow{o} J) \leftrightarrow (J \xleftarrow{o} I) \\
\phi_{\text{dual}_s} &= \forall I,\ \forall J,\ (I \xrightarrow{s} J) \leftrightarrow (J \xleftarrow{s} I) \\
\phi_{\text{dual}_f} &= \forall I,\ \forall J,\ (I \xrightarrow{f} J) \leftrightarrow (J \xleftarrow{f} I) \\
\phi_{\text{dual}_d} &= \forall I,\ \forall J,\ (I \xrightarrow{d} J) \leftrightarrow (J \xleftarrow{d} I)
\end{aligned}$$

And finally, the transitivity of the different relations is given by the following schema, where we range i, j over I and use $T(i, j)$ to denote the entry under row i and column j of Table 2.

$$\phi_{\text{trans}_{i,j}} = \forall I,\ \forall J,\ \forall K,\ (I \xrightarrow{i} J) \wedge (J \xrightarrow{j} K) \rightarrow (I \xrightarrow{T(i,j)} J)$$

With this, we can now give the theory of interval algebras as the union of all these sentences.

Definition 3.2. We define the theory of Allen interval algebras $\mathbb{T}_{\text{AIA}}$ as

$$\mathbb{T}_{\text{AIA}} = \{\phi_{\text{exh}}, \phi_{\text{mutex}}, \phi_{\text{dual}_>}, \phi_{\text{dual}_m}, \phi_{\text{dual}_o}, \phi_{\text{dual}_s}, \phi_{\text{dual}_f}, \phi_{\text{dual}_d}\} \cup \{\phi_{\text{trans}_{i,j}} \mid i, j \in I\}$$

With a theory defined, the first thing to determine is whether or not it is satisfiable, since an inconsistent theory would not be very interesting. We should have as models at least the models considered in Allen's original paper, which we now check.

Definition 3.3. Given a linear order L we define its set of non-zero intervals $\text{Int}\,(P)$ as the set

$$\text{Int}\,(L) = \{(x_1, x_2) \mid x_1 < x_2\} \subseteq L^2$$

We can turn this into a $\mathcal{L}_{\text{AIA}}$-structure under the interpretations:

- $(x_1, x_2) \xrightarrow{i} (y_1, y_2)$ if and only if $L \models \phi_{\xrightarrow{i}}(x_1, x_2, y_1, y_2)$.
- $(x_1, x_2) \xleftarrow{i} (y_1, y_2)$ if and only if $L \models \phi_{\xrightarrow{i}}(y_1, y_2, x_1, x_2)$.

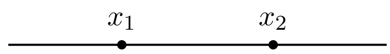
where we range i over the indexing set $\{<, m, o, s, f, d\}$ and define the first order $\mathcal{L}_{\text{SLO}}$-formulas by

$$\begin{aligned}
\phi_{\xrightarrow{<}}(x_1, x_2, y_1, y_2) &= (x_1 < x_2) \wedge (x_2 < y_1) \wedge (y_1 < y_2) \\
\phi_{\xrightarrow{m}}(x_1, x_2, y_1, y_2) &= (x_1 < x_2) \wedge (x_2 = y_1) \wedge (y_1 < y_2) \\
\phi_{\xrightarrow{o}}(x_1, x_2, y_1, y_2) &= (x_1 < y_1) \wedge (y_1 < x_2) \wedge (x_2 < y_2) \\
\phi_{\xrightarrow{s}}(x_1, x_2, y_1, y_2) &= (x_1 = y_1) \wedge (y_1 < x_2) \wedge (x_2 < y_2) \\
\phi_{\xrightarrow{f}}(x_1, x_2, y_1, y_2) &= (y_1 < x_1) \wedge (x_1 < x_2) \wedge (x_2 = y_2) \\
\phi_{\xrightarrow{d}}(x_1, x_2, y_1, y_2) &= (y_1 < x_1) \wedge (x_1 < x_2) \wedge (x_2 < y_2)
\end{aligned}$$

Notice that the subset $\text{Int}\,(L) \subseteq L^2$ is definable using the language of strict linear orders. Similarly, the interpretations of the relations are also given by formulas in this language. As a result, the structure $\text{Int}\,(L)$ is interpretable in the strict linear order L.

Theorem 3.4. *Given a strict linear order L, $Int(L)$ is a model of $\mathbb{T}_{AIA}$ under the above interpretations.*

Proof. First we check that $\text{Int}\,(L) \models \phi_{\text{exh}}$. We want to count the number of possible arrangements of $x_1, x_2, y_1, y_2 \in L$ under the restriction that $x_1 < x_2$ and $y_1 < y_2$. Since we need $x_1 < x_2$, we can start with just the following



Then there are 5 possible positions for y_1: we can have $y_1 = x_i$ for some i or y_1 can lie on one of the lines, distinct from both x_1 and x_2. Then, since we need $y_1 < y_2$, the number of possible arrangements can be seen to be $5+5+3+3+1 = 13$. Including equality, we have 13 relation symbols, each representing a different ordering between the x_i and y_i, which must mean that our relation symbols are exhaustive after all.

Next we check that the relation symbols are mutually exclusive. This comes from the fact that each relation symbol encodes a separate possible ordering between the interval start and end points. As it is not possible to have 2 different orderings of the same fixed elements, the relation symbols must be mutually exclusive. For example, suppose that we have

$$(x_1, x_2) \xrightarrow{<} (y_1, y_2) \qquad\qquad (x_1, x_2) \xrightarrow{o} (y_1, y_2)$$

This happens only if we have both

$$x_1 < x_2 < y_1 < y_2 \qquad\qquad x_1 < y_1 < x_2 < y_2$$

and so $x_2 < y_1 < x_2$. Since $<$ is irreflexive, this cannot happen.

Checking transitivity is quite routine, the main difficulty being the sheer amount of cases. We will explicitly check the upper left quadrant in Table 2 and leave the rest for the diligent reader. Fix $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathrm{Int}\,(L)$, which we will denote by x, y, z respectively, then:

- Suppose that $x \xrightarrow{<} y$, so $x_1 < x_2 < y_1 < y_2$. If $y \xrightarrow{<mos} z$ then in all cases we must have $y_1 \leq z_1$, hence $x_1 < x_2 < z_1 < z_2$ and $x \xrightarrow{<} z$. On the other hand, if $y \xrightarrow{fd} z$ then we have $z_1 < y_1 \leq z_2$. This means that $x_2 < z_2$, restricting the possible relations between x and z to $x \xrightarrow{<mosd} z$ as expected.
- Suppose that $x \xrightarrow{m} y$ so $x_1 < x_2 = y_1 < y_2$. If $y \xrightarrow{<mo} z$ then in all cases we must have $y_1 < z_1$ implying that $x \xrightarrow{<} z$. If $y \xrightarrow{s} z$ then $y_1 = z_1 < z_2$ and $x \xrightarrow{m} z$ too. Finally, if $y \xrightarrow{fd} z$ then we have $z_1 < y_1 < y_2 \leq z_2$, but $x_2 = y_1$ implying that $x \xrightarrow{osd} z$.
- Suppose that $x \xrightarrow{o} y$, in which case we have $x_1 < y_1 < x_2 < y_2$. Then if $y \xrightarrow{<m} z$ we see that $x_1 < x_2 < z_1 < z_2$ so $x \xrightarrow{<} z$. If $y \xrightarrow{o} z$ then $x_1 < z_1$ and $x_2 < z_2$, leaving $x \xrightarrow{<mo} z$ as the only options. If $y \xrightarrow{s} z$ then $x_1 < y_1 = z_1 < x_2 < y_2 < z_2$ so $x \xrightarrow{o} z$ too. On the other hand if $y \xrightarrow{fd} z$ then $z_1 < x_2 < z_2$ and hence $x \xrightarrow{osd} z$.
- Suppose that $x \xrightarrow{s} y$ so $x_1 = y_1 < x_2 < y_2$. In the case that $y \xrightarrow{<m} z$ then $x_1 < x_2 < z_1 < z_2$, ie. $x \xrightarrow{<} z$. Similarly to before, if $y \xrightarrow{o} z$ then $x_1 < z_2$ and $x_2 < z_2$, so $x \xrightarrow{<mo} z$. If $y \xrightarrow{s} z$ then we get $x_1 = y_1 = z_1 < x_2 < y_2 < z_2$ so $x \xrightarrow{s} z$ too. For the last case, if $y \xrightarrow{fd} z$ then $z_1 < y_1 = x_1 < x_1 < y_2 \leq z_2$ so $x \xrightarrow{d} z$.

- Suppose that $x \xrightarrow{f} y$, then $y_1 < x_1 < x_2 = y_2$. If $y \xrightarrow{<} z$ then we get $x_1 < x_2 = y_2 < z_1 < z_2$ and so $x \xrightarrow{<} z$. If $y \xrightarrow{m} z$ then similarly we have $x_1 < x_2 = y_2 = z_1 < z_2$ so $x \xrightarrow{m} z$. In the case that $y \xrightarrow{o} z$ we can see that $z_1 < y_2 = x_2 < z_2$ so $x \xrightarrow{osd} z$. If $x \xrightarrow{sd} z$ then $z_1 < y_1 = x_1 < x_2 < y_2 \leq z_2$, implying $x \xrightarrow{d} z$. Finally, if $y \xrightarrow{f} z$ then $z_1 < y_1 < x_1 < x_2 = y_2 = z_2$ and $x \xrightarrow{f} z$.
- Suppose that $x \xrightarrow{d} y$ so $y_1 < x_1 < x_2 < y_2$. If $y \xrightarrow{<m} z$ then we have $x_1 < x_1 < y_2 \leq z_1 < z_2$ so $x \xrightarrow{<} z$. If $y \xrightarrow{o} z$ then all we can say is that $x_2 < z_2$, leaving us with $x \xrightarrow{<mosd} z$. For the final case, if $y \xrightarrow{sfd} z$ then we end up with $z_1 \leq y_1 < x_1 < x_2 < y_2 \leq z_2$, so $x \xrightarrow{d} z$.

The rest of the cases can be checked similarly.

Finally, the different duality formulas hold by definition, so $\mathrm{Int}\,(L) \models \mathbb{T}_{\mathrm{AIA}}$. □

Corollary 3.5. *Allen's interval algebras are satisfiable*

So the theory is satisfiable and it includes the main class of models we are interested in. Since the theory is relational and universal, it also means that we can pick out any subset of intervals over a linear order and that will give us another model of $\mathbb{T}_{\mathrm{AIA}}$, which agrees with our intuition for interval algebras. We will also see later that every model of $\mathbb{T}_{\mathrm{AIA}}$ can be seen as a substructure of $\mathrm{Int}\,(L)$ for some linear order L, which bodes very well for our axiomatisation. But before we can see this, we need to develop some more machinery about interval algebras.

Consider the function $\phi_\sim : \{0,1\}^2 \to \{\mathcal{L}_{\mathrm{AIA}} - \text{formulas } \phi(I,J)\}$ defined by

$$\begin{aligned}
\phi_\sim(0,0)(I,J) &= (I \xrightarrow{s} J) \vee (I \xleftarrow{s} J) \vee (I = J)\\
\phi_\sim(0,1)(I,J) &= (I \xleftarrow{m} J)\\
\phi_\sim(1,0)(I,J) &= (I \xrightarrow{m} J)\\
\phi_\sim(1,1)(I,J) &= (I \xrightarrow{f} J) \vee (I \xleftarrow{f} J) \vee (I = J)
\end{aligned}$$

This function takes as inputs tuples (n,m) with $n,m \in \{0,1\}$ and assigns them the $\mathcal{L}_{\mathrm{AIA}}$-formulas $\phi_\sim(n,m)$ with I and J as free variables. We want to use this function to quotient out the disjoint union $A + A$ in the following way:

$$(n,I) \sim (m,J) \iff A \models \phi_\sim(n,m)(I,J)$$

Of course, first we need to figure out whether this gives an equivalence relation.

Proposition 3.6. *The relation $\sim$ defined as above is an equivalence relation on $A + A$.*

Proof. The relation is reflexive, since $\phi_\sim(n,n)$ always contains $I = J$ as a disjunct, hence $A \models \phi_\sim(n,n)(I,I)$ and $(n,I) \sim (n,I)$.

The relation is also symmetric, since $A \models \phi_\sim(n,m)(I,J)$ if and only if $A \models \phi_\sim(m,n)(J,I)$. To see this, notice that the formula $\phi_\sim(m,n)$ is given by taking the dual of the relation symbols in $\phi_\sim(n,m)$. Hence $(n,I) \sim (m,J)$ if and only if $(m,J) \sim (n,I)$.

Showing the relation is transitive requires going through the 8 cases based on the possible values of the triple $(n,m,l) \in \{0,1\}^3$ and showing that in each case, if $A \models \phi_\sim(n,m)(I,J)$ and $A \models \phi_\sim(m,l)(J,K)$ then $A \models \phi_\sim(n,l)(I,K)$. This can be done by using the transitivity axioms of interval algebras and considering every case. We go through a more complicated case in the proof of Theorem 3.8, where the process is completely analogous, hence we skip the proof here for brevity. □

Definition 3.7. Given an Allen interval algebra A, we define the points of A as

$$\mathrm{Pts}\,(A) = \frac{A+A}{\sim}$$

Consider the function $\phi_< : \{0,1\}^2 \to \{\mathcal{L}_{\mathrm{AIA}} - \text{formulas } \phi(I,J)\}$ defined by

$$\begin{aligned}
\phi_<(0,0)(I,J) &= (I \xrightarrow{<} J) \vee (I \xrightarrow{m} J) \vee (I \xrightarrow{o} J) \vee (I \xleftarrow{f} J) \vee (I \xleftarrow{d} J)\\
\phi_<(0,1)(I,J) &= \neg\,(I \xleftarrow{<} J) \wedge \neg\,(I \xleftarrow{m} J)\\
\phi_<(1,0)(I,J) &= (I \xrightarrow{<} J)\\
\phi_<(1,1)(I,J) &= (I \xrightarrow{<} J) \vee (I \xrightarrow{m} J) \vee (I \xrightarrow{o} J) \vee (I \xrightarrow{s} J) \vee (I \xrightarrow{d} J)
\end{aligned}$$

Using this, we may order the start and end points of intervals in A, turning $\mathrm{Pts}\,(A)$ into a strict linear order.

Theorem 3.8. *Given an Allen interval algebra A, the interpretation of the symbol $<$ in $Pts\,(A)$ given by $[(n,I)] < [(m,J)] \iff A \models \phi_<(n,m)(I,J)$ is well-defined and turns A into a model of $\mathbb{T}_{SLO}$.*

Proof. First we must check that this ordering relation on $\mathrm{Pts}\,(A)$ is well-defined. For this fix intervals $I_1, I_2, J_1, J_2 \in A$ and $n_1, n_2, m_1, m_2 \in \{0,1\}$ and assume that

$$[(n_1,I_1)] = [(n_2,I_2)] < [(m_2,J_2)] = [(m_1,J_1)]$$

Then we must show that indeed, $[(n_1,I_1)] < [(m_1,J_1)]$. There are 16 distinct cases which we cover in Table 3. In each of the cases, the possible relations $I_1 \longrightarrow J_1$ given in the last column imply that $A \models \phi_<(n_1,m_1)(I_1,J_1)$, so $<$ is well-defined.

Now that we know that our definition of $<$ does not depend on a choice of representative, we need to check whether it satisfies $\mathbb{T}_{\mathrm{SLO}}$:

- **irreflexivity**: Fix some element $[(n,I)] \in \mathrm{Pts}\,(A)$. Since the interval algebra relations (along with equality) are mutually exclusive and $I = I$, no other relation can hold for the pair (I,I). Regardless of the value of n, $\phi_<(n,n)(I,I)$ does not include $I = I$ as a disjunct, so $A \not\models \phi_<(n,n)(I,I)$ and $<$ is irreflexive.

(n_1, n_2, m_1, m_2)	Relation between I_1, I_2, J_1, J_2	Relation between I_1, J_1
$(0,0,0,0)$	$I_1 \xrightarrow{s=S} I_2 \xrightarrow{<moFD} J_2 \xrightarrow{s=S} I_1$	$I_1 \xrightarrow{<moFD} J_1$
$(0,0,0,1)$	$I_1 \xrightarrow{s=S} I_2 \xrightarrow{<mosfd=OSFD} J_2 \xrightarrow{m} I_1$	$I_1 \xrightarrow{<moFD} J_1$
$(0,0,1,0)$	$I_1 \xrightarrow{s=S} I_2 \xrightarrow{<moFD} J_2 \xrightarrow{M} I_1$	$I_1 \xrightarrow{<mosfd=OSFD} J_1$
$(0,0,1,1)$	$I_1 \xrightarrow{s=S} I_2 \xrightarrow{<mosfd=OSFD} J_2 \xrightarrow{f=F} I_1$	$I_1 \xrightarrow{<mosfd=OSFD} J_1$
$(0,1,0,0)$	$I_1 \xrightarrow{M} I_2 \xrightarrow{<} J_2 \xrightarrow{s=S} I_1$	$I_1 \xrightarrow{<moFD} J_1$
$(0,1,0,1)$	$I_1 \xrightarrow{M} I_2 \xrightarrow{<mosd} J_2 \xrightarrow{m} I_1$	$I_1 \xrightarrow{<moFD} J_1$
$(0,1,1,0)$	$I_1 \xrightarrow{M} I_2 \xrightarrow{<} J_2 \xrightarrow{M} I_1$	$I_1 \xrightarrow{<mosfd=OSFD} J_1$
$(0,1,1,1)$	$I_1 \xrightarrow{M} I_2 \xrightarrow{<mosd} J_2 \xrightarrow{f=F} I_1$	$I_1 \xrightarrow{<mosfd=OSFD} J_1$
$(1,0,0,0)$	$I_1 \xrightarrow{m} I_2 \xrightarrow{<moFD} J_2 \xrightarrow{s=S} I_1$	$I_1 \xrightarrow{<} J_1$
$(1,0,0,1)$	$I_1 \xrightarrow{m} I_2 \xrightarrow{<mosfd=OSFD} J_2 \xrightarrow{m} I_1$	$I_1 \xrightarrow{<} J_1$
$(1,0,1,0)$	$I_1 \xrightarrow{m} I_2 \xrightarrow{<moFD} J_2 \xrightarrow{M} I_1$	$I_1 \xrightarrow{<mosd} J_1$
$(1,0,1,1)$	$I_1 \xrightarrow{m} I_2 \xrightarrow{<mosfd=OSFD} J_2 \xrightarrow{f=F} I_1$	$I_1 \xrightarrow{<mosd} J_1$
$(1,1,0,0)$	$I_1 \xrightarrow{f=F} I_2 \xrightarrow{<} J_2 \xrightarrow{s=S} I_1$	$I_1 \xrightarrow{<} J_1$
$(1,1,0,1)$	$I_1 \xrightarrow{f=F} I_2 \xrightarrow{<mosd} J_2 \xrightarrow{m} I_1$	$I_1 \xrightarrow{<} J_1$
$(1,1,1,0)$	$I_1 \xrightarrow{f=F} I_2 \xrightarrow{<} J_2 \xrightarrow{M} I_1$	$I_1 \xrightarrow{<mosd} J_1$
$(1,1,1,1)$	$I_1 \xrightarrow{f=F} I_2 \xrightarrow{<mosd} J_2 \xrightarrow{f=F} I_1$	$I_1 \xrightarrow{<mosd} J_1$

Table 3: All cases to check if the ordering on $\mathrm{Pts}\,(A)$ is well defined.

- **transitivity**: To show transitivity we start by fixing intervals $I, J, K \in A$ and assuming that $[(n, I)] < [(m, J)]$ and $[(m, J)] < [(l, K)]$. Then there are 8 cases based on the values of $n, m, k \in \{0, 1\}$, all of which are considered in Table 4 up to Table 11.
- **trichotomy**: Fix two $I, J \in A$. To check the trichotomy condition holds, we need to show that for all $n, m \in \{0, 1\}$ we have

$$A \models \phi_<(n,m)(I,J) \vee \phi_\sim(n,m)(I,J) \vee \phi_<(m,n)(J,I)$$

There are 3 cases that we must deal with separately to show this:

  - First we consider $[(0, I)]$ and $[(0, J)]$. The formula $\phi_<(0,0)(I,J)$ is a disjunction 5 of our basic relations and $\phi_<(0,0)(J,I)$ the disjunction of its duals. Then $\phi_\sim(0,0)(I,J)$ is a disjunction of the 2 missing relations and equality. Hence by exhaustiveness of the interval algebra relations and equality, it must be the case that $A \models \phi_<(0,0)(I,J) \vee \phi_\sim(0,0)(I,J) \vee \phi_<(0,0)(J,I)$
  - Next, we consider $[(0, I)]$ and $[(1, J)]$. Notice that the formula $\phi_<(0,1)(I,J)$ is

the negation of $\phi_\sim(0,1)(I,J) \vee \phi_<(1,0)(J,I)$, hence by the law of the excluded middle, the triple disjunction must hold.

– Finally we consider $[(1,I)]$ and $[(1,J)]$. The situation here is similar to the $n = m = 0$ case, since $\phi_<(1,1)(I,J)$ is a disjunction of 5 basic relations, then $\phi_<(1,1)(J,I)$ is the disjunction of its duals and $peq(1,1)(I,J)$ is the disjunction of the 2 missing relations plus equality.

When $(n,m) = (1,0)$, since $\sim$ is symmetric, we know that

$$A \models \phi_\sim(1,0)(I,J) \iff A \models \phi_\sim(0,1)(J,I)$$

which allows us to reduce to the second case above.

□

In general, this will not be interpretable in the interval algebra A, since we need to take the disjoint union of two sets which is not a priori available in first order logic. However, provided that $|A| \neq 1$, then it is possible to define quotient A^2 by an appropriate definable equivalence relation, something along the lines of

$$\phi(I,J) = \neg(I = J)$$

So then we would encode $(0,I) \in A + A$ as the pair $(I,I) \in A^2$ and $(1,I) \in A + A$ as $(I,J) \in A^2$ where J is any interval different from I. Modifying the above definitions to use this encoding would then show that the strict linear order $\mathrm{Int}\,(A)$ can be interpreted in A.

$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$	o	$<$	$<$	D	$<$	$< moFD$
$<$	m	$<$	o	m	$<$	D	m	oFD
$<$	o	$<$	o	o	$< mo$	D	o	oFD
$<$	F	$<$	o	F	$< mo$	D	F	D
$<$	D	$<$	o	D	$< moFD$	D	D	D
m	$<$	$<$	F	$<$	$<$			
m	m	$<$	F	m	m			
m	o	$<$	F	o	o			
m	F	$<$	F	F	F			
m	D	$<$	F	D	D			

Table 4: The transitivity table for the $[(0, I)] < [(0, J)]$ and $[(0, J)] < [(0, K)]$ case. For this to imply $[(0, I)] < [(0, K)]$ we need the $I \to K$ columns to all contain a subset of the string $< moFD$.

$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$	o	$<$	$<$	D	$<$	$< moFD$
$<$	m	$<$	o	m	$<$	D	m	oFD
$<$	o	$<$	o	o	$< mo$	D	o	oFD
$<$	s	$<$	o	s	o	D	s	oFD
$<$	f	$< mosd$	o	f	osd	D	f	OSD
$<$	d	$< mosd$	o	d	osd	D	d	concur
$<$	$=$	$<$	o	$=$	o	D	$=$	D
$<$	O	$< mosd$	o	O	concur	D	O	OSD
$<$	S	$<$	o	S	oFD	D	S	D
$<$	F	$<$	o	F	$< mo$	D	F	D
$<$	D	$<$	o	D	$< moFD$	D	D	D
m	$<$	$<$	F	$<$	$<$			
m	m	$<$	F	m	m			
m	o	$<$	F	o	o			
m	s	m	F	s	o			
m	f	osd	F	f	$f = F$			
m	d	osd	F	d	osd			
m	$=$	m	F	$=$	F			
m	O	osd	F	O	OSD			
m	S	m	F	S	D			
m	F	$<$	F	F	F			
m	D	$<$	F	D	D			

Table 5: The transitivity table for the $[(0,I)] < [(0,J)]$ and $[(0,J)] < [(1,K)]$ case. For this to imply $[(0,I)] < [(1,K)]$ we need the $I \to K$ columns to all contain a subset of the string $< mosfd = OSFD$. Recall that concur is shorthand for $osfd = OSFD$

$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$
m	$<$	$<$
o	$<$	$<$
s	$<$	$<$
f	$<$	$<$
d	$<$	$<$
$=$	$<$	$<$
O	$<$	$< moFD$
S	$<$	$< moFD$
F	$<$	$<$
D	$<$	$< moFD$

Table 6: The transitivity table for the $[(0, I)] < [(1, J)]$ and $[(1, J)] < [(0, K)]$ case. For this to imply $[(0, I)] < [(0, K)]$ we need the $I \to K$ columns to all contain a subset of the string $< mofD$.

$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$	f	$<$	$<$	S	$<$	$< moFD$
$<$	m	$<$	f	m	m	S	m	oFD
$<$	o	$<$	f	o	osd	S	o	oFD
$<$	s	$<$	f	s	d	S	s	$s = S$
$<$	d	$< mosd$	f	d	d	S	d	fdO
m	$<$	$<$	d	$<$	$<$	F	$<$	$<$
m	m	$<$	d	m	$<$	F	m	m
m	o	$<$	d	o	$< mosd$	F	o	o
m	s	m	d	s	d	F	s	o
m	d	osd	d	d	d	F	d	osd
o	$<$	$<$	$=$	$<$	$<$	D	$<$	$< moFD$
o	m	$<$	$=$	m	m	D	m	oFD
o	o	$< mo$	$=$	o	o	D	o	oFD
o	s	o	$=$	s	s	D	s	oFD
o	d	osd	$=$	d	d	D	d	concur
s	$<$	$<$	O	$<$	$< moFD$			
s	m	$<$	O	m	oFD			
s	o	$< mo$	O	o	concur			
s	s	s	O	s	fdO			
s	d	d	O	d	fdO			

Table 7: The transitivity table for the $[(0, I)] < [(1, J)]$ and $[(1, J)] < [(1, K)]$ case. For this to imply $[(0, I)] < [(1, K)]$ we need the $I \to K$ columns to all contain a subset of the string $< mosfd = OSFD$. Recall that concur is shorthand for $osfd = OSFD$.

$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$
$<$	m	$<$
$<$	o	$<$
$<$	F	$<$
$<$	D	$<$

Table 8: The transitivity table for the $[(1, I)] < [(0, J)]$ and $[(0, J)] < [(0, K)]$ case. For this to imply $[(1, I)] < [(0, K)]$ we need the $I \to K$ columns to all contain $<$.

$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$
$<$	m	$<$
$<$	o	$<$
$<$	s	$<$
$<$	f	$< mosd$
$<$	d	$< mosd$
$<$	$=$	$<$
$<$	O	$< mosd$
$<$	S	$<$
$<$	F	$<$
$<$	D	$<$

Table 9: The transitivity table for the $[(1, I)] < [(0, J)]$ and $[(0, J)] < [(1, K)]$ case. For this to imply $[(1, I)] < [(1, K)]$ we need the $I \to K$ columns to all contain a subset of the string $< mosd$.

$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$
m	$<$	$<$
o	$<$	$<$
s	$<$	$<$
d	$<$	$<$

Table 10: The transitivity table for the $[(1, I)] < [(1, J)]$ and $[(1, J)] < [(0, K)]$ case. For this to imply $[(1, I)] < [(0, K)]$ we need the $I \to K$ columns to all contain $<$.

$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$	$I \to J$	$J \to K$	$I \to K$
$<$	$<$	$<$	o	$<$	$<$	d	$<$	$<$
$<$	m	$<$	o	m	$<$	d	m	$<$
$<$	o	$<$	o	o	$< mo$	d	o	$< mosd$
$<$	s	$<$	o	s	o	d	s	d
$<$	d	$< mosd$	o	d	osd	d	d	d
m	$<$	$<$	s	$<$	$<$			
m	m	$<$	s	m	$<$			
m	o	$<$	s	o	$< mo$			
m	s	m	s	s	s			
m	d	osd	s	d	d			

Table 11: The transitivity table for the $[(1, I)] < [(1, J)]$ and $[(1, J)] < [(1, K)]$ case. For this to imply $[(1, I)] < [(1, K)]$ we need the $I \to K$ columns to all contain a subset of the string $< mosd$.

4 The Points-Intervals Adjunction

Definition 4.1. Given a theory $\mathbb{T}$ over language $\mathcal{L}$, we denote by $\mathrm{Mod}(\mathcal{L}, \mathbb{T})$ the category with objects the models of $\mathbb{T}$ and arrows the $\mathcal{L}$-embeddings.

Remark. For brevity, we introduce the notation:

$$\mathbf{SLO} := \mathrm{Mod}(\mathcal{L}_{\mathrm{SLO}}, \mathbb{T}_{\mathrm{SLO}}) \text{ and } \mathbf{AIA} := \mathrm{Mod}(\mathcal{L}_{\mathrm{AIA}}, \mathbb{T}_{\mathrm{AIA}})$$

The Yoneda point of view from category theory tells us that the action of a functor on maps is equally as important, if not more, than its action on objects. The next two theorems will tell us how to extend the interval and points constructions to acting on maps.

Theorem 4.2. *We can turn $Int(-)$ into a functor*

$$Int(-) : \boldsymbol{SLO} \to \boldsymbol{AIA}$$

by sending arrows $f : M \to N$ in $\boldsymbol{SLO}$ to $Int(f) : Int(M) \to Int(N)$ defined by

$$Int(f)(x_1, x_2) = (f(x_1), f(x_2))$$

Proof. First we show that for a $\mathcal{L}_{\mathrm{SLO}}$-embedding $f : L \to M$ between strict linear orders L, M, the mapping $\mathrm{Int}(f)$ gives a an $\mathcal{L}_{\mathrm{AIA}}$-embedding. Consider the relations in $\mathcal{L}_{\mathrm{AIA}}$, and their interpretations in $\mathrm{Int}(L)$: all of the relations are defined by quantifier-free $\mathcal{L}_{\mathrm{SLO}}$-formulas, whose truth value must be preserved under $\mathcal{L}_{\mathrm{SLO}}$ embeddings like f. Since $\mathrm{Int}(f)$ simply applies f pointwise, $\mathrm{Int}(f)$ must preserve the truth value of the relation symbols in $\mathcal{L}_{\mathrm{AIA}}$, in other words, it is an $\mathcal{L}_{\mathrm{AIA}}$-embedding.

Now we just need to check that $\mathrm{Int}(-)$ satisfies the two functor axioms:

- **preserves identity arrows**: Fix some strict linear order L and some interval $(x, y) \in \mathrm{Int}(L)$, then
$$\mathrm{Int}(\mathrm{id}_L)(x, y) = (\mathrm{id}_L(x), \mathrm{id}_L(y)) = (x, y)$$
- **respects arrow composition**: Fix three strict linear orders L, M, N along with arrows $f : L \to M$, $g : M \to N$ and some interval $(x, y) \in \mathrm{Int}(L)$, then
$$\begin{aligned}\mathrm{Int}(g \circ f)(x, y) &= (g \circ f(x), g \circ f(y)) \\ &= (g(f(x)), g(f(y))) \\ &= \mathrm{Int}(g)(\mathrm{Int}(f)(x, y)) \\ &= \mathrm{Int}(g) \circ \mathrm{Int}(f)(x, y)\end{aligned}$$

□

Theorem 4.3. *We can turn Pts $(-)$ into a functor*

$$\mathit{Pts}\,(-) : \boldsymbol{AIA} \to \boldsymbol{SLO}$$

by sending arrows $f : A \to B$ in $\boldsymbol{AIA}$ to $\mathit{Pts}\,(f) : \mathit{Pts}\,(A) \to \mathit{Pts}\,(B)$ defined by

$$\mathit{Pts}\,(f)\,(0, I) = (0, f(I)) \qquad \text{and} \qquad \mathit{Pts}\,(f)\,(1, I) = (1, f(I))$$

Proof. Given an arrow $f : A \to B$, we must check that $\mathrm{Pts}\,(f)$ is a well defined map, and that it is an $\mathcal{L}_{\mathrm{SLO}}$-embedding. These facts both follow by noticing that f is an $\mathcal{L}_{\mathrm{AIA}}$-embedding, so it preserves the truth of quantifier-free $\mathcal{L}_{\mathrm{AIA}}$-formulas, so

$$\begin{aligned}(n, I) \sim (m, J) &\iff A \models \phi_{\sim}(n, m)(I, J) \\ &\iff A \models \phi_{\sim}(n, m)(f(I), f(J)) \\ &\iff (n, f(I)) \sim (m, f(J))\end{aligned}$$

and similarly

$$\begin{aligned}[(n, I)] < [(m, J)] &\iff A \models \phi_{<}(n, m)(I, J) \\ &\iff A \models \phi_{<}(n, m)(f(I), f(J)) \\ &\iff \mathrm{Pts}\,(f)\,([(n, I)]) < \mathrm{Pts}\,(f)\,([(m, J)])\end{aligned}$$

for any two $(n, I), (m, J) \in A + A$, since $\phi_{\sim}(n, m)$ and $\phi_{<}(n, m)$ are always quantifier-free.

Next, to see that $\mathrm{Pts}\,(-)$ satisfies the functor axioms:

- **preserves identity arrows**: Fix some interval algebra A and some element $[(n, I)] \in \mathrm{Pts}\,(A)$. Then notice that

  $$\mathrm{Pts}\,(\mathrm{id}_A)\,([(n, I)]) = [(n, \mathrm{id}_A(I))] = [(n, I)] = \mathrm{id}_{\mathrm{Pts}(A)}([(n, I)])$$

  Hence $\mathrm{Pts}\,(\mathrm{id}_A) = \mathrm{id}_{\mathrm{Pts}(A)}$.

- **respects arrow composition**: Fix interval algebras A, B, C along with arrows $f : A \to B$, $g : B \to C$. Then for all elements $[(n, I)] \in \mathrm{Pts}\,(A)$:

  $$\begin{aligned}\mathrm{Pts}\,(g \circ f)\,([(n, I)]) &= [(n, g \circ f(I))] \\ &= [(n, g(f(I)))] \\ &= \mathrm{Pts}\,(g)\,(\mathrm{Pts}\,(f)\,([n, I])) \\ &= \mathrm{Pts}\,(g) \circ \mathrm{Pts}\,(f)\,([n, I])\end{aligned}$$

  Hence $\mathrm{Pts}\,(g \circ f) = \mathrm{Pts}\,(g) \circ \mathrm{Pts}\,(f)$ as expected.

$\square$

Remark. From now on, given an interval algebra A and interval $I \in A$, we will use $I_- := [(0, I)]$ and $I_+ := [(1, I)]$ to refer to the respective elements of $\mathrm{Pts}\,(A)$.

As we will see now, these two functors are adjoints, which indicates a very special and close connection between strict linear orders and interval algebras.

Theorem 4.4. *Pts$(-)$ is left adjoint to Int$(-)$.*

Proof. We will prove this through the Hom-Set definition of an adjunction, so we wish to find an isomorphism $\mathbf{SLO}(\mathrm{Pts}(A), L) \cong \mathbf{AIA}(A, \mathrm{Int}(L))$ which is natural in both A and L.

We start by defining the forward map

$$\phi_{A,L} : \mathbf{SLO}(\mathrm{Pts}(A), L) \to \mathbf{AIA}(A, \mathrm{Int}(L))$$

which sends an $\mathcal{L}_{\mathrm{SLO}}$-embedding $f : \mathrm{Pts}(A) \to L$ to the $\mathcal{L}_{\mathrm{AIA}}$-embedding

$$\phi_{A,L}(f) : A \to \mathrm{Int}(L) \quad \text{sending} \quad I \mapsto (f(I_-), f(I_+))$$

To see this is a $\mathcal{L}_{\mathrm{AIA}}$-embedding, by duality, it suffices to consider the $\xrightarrow{i}$ relations for $i \in \{<, m, o, s, f, d\}$, so we fix two intervals $I, J \in A$ and then:

$$\begin{aligned}
I \xrightarrow{<} J &\iff I_+ < J_- \\
&\iff f(I_+) < f(J_-) \\
&\iff \phi_{A,L}(f)(I) \xrightarrow{<} \phi_{A,L}(f)(J) \\
I \xrightarrow{m} J &\iff I_+ = J_- \\
&\iff f(I_+) = f(J_-) \\
&\iff \phi_{A,L}(f)(I) \xrightarrow{m} \phi_{A,L}(f)(J) \\
I \xrightarrow{o} J &\iff I_- < J_- < I_+ < J_+ \\
&\iff f(I_-) < f(J_-) < f(I_+) < f(J_+) \\
&\iff \phi_{A,L}(f)(I) \xrightarrow{o} \phi_{A,L}(f)(J)
\end{aligned}
\qquad
\begin{aligned}
I \xrightarrow{s} J &\iff I_- = J_- < I_+ < J_+ \\
&\iff f(I_-) = f(J_-) < f(I_+) < f(J_+) \\
&\iff \phi_{A,L}(f)(I) \xrightarrow{s} \phi_{A,L}(f)(J) \\
I \xrightarrow{f} J &\iff J_- < I_- < I_+ = J_+ \\
&\iff f(J_-) < f(I_-) < f(I_+) = f(J_+) \\
&\iff \phi_{A,L}(f)(I) \xrightarrow{f} \phi_{A,L}(f)(J) \\
I \xrightarrow{d} J &\iff J_- < I_- < I_+ < J_+ \\
&\iff f(J_-) < f(I_-) < f(I_+) < f(J_+) \\
&\iff \phi_{A,L}(f)(I) \xrightarrow{d} \phi_{A,L}(f)(J)
\end{aligned}$$

Next, we define the inverse map

$$\psi_{A,L} : \mathbf{AIA}(A, \mathrm{Int}(L)) \to \mathbf{SLO}(\mathrm{Pts}(A), L)$$

which sends an $\mathcal{L}_{\mathrm{AIA}}$-embedding $g : A \to \mathrm{Int}(L)$ to the $\mathcal{L}_{\mathrm{SLO}}$-embedding given by

$$\psi_{A,L}(f) : \mathrm{Pts}(A) \to L \quad \text{sending} \quad (n, I) \mapsto (\text{let } (x_0, x_1) := g(I) \text{ in } x_n)$$

Now, we need to ensure these maps are well defined and that they are indeed $\mathcal{L}_{\mathrm{SLO}}$-embeddings. To check that $\psi_{A,L}(f)$ is well defined, we fix intervals $I, J \in A$ and get three cases to consider. In all of these cases we let $f(I) = (a, b)$ and $f(J) = (c, d)$:

- If $(0, I) \sim (0, J)$ then we must have

$$A \models (I \xrightarrow{s} J) \vee (I \xleftarrow{s} J) \vee (I = J)$$

Since f is an $\mathcal{L}_{\mathrm{AIA}}$-embedding, this means that

$$\mathrm{Int}\,(L) \models (f(I) \xrightarrow{s} f(J)) \vee (f(I) \xleftarrow{s} f(J)) \vee (f(I) = f(J))$$

Equivalently, the above says that $a = c$, so $\psi_{A,L}(f)(I_-) = a = c = \psi_{A,L}(f)(J_-)$.

- If $(0, I) \sim (1, J)$ then the following is true

$$A \models I \xleftarrow{m} J$$

Which implies that

$$\mathrm{Int}\,(L) \models f(I) \xleftarrow{m} f(J)$$

And this happens exactly when $a = d$, meaning $\psi_{A,L}(f)(I_-) = a = d = \psi_{A,L}(f)(J_+)$. By symmetry of our equivalence relation this also deals with the $(1, I) \sim (0, J)$ case.

- If $(1, I) \sim (1, J)$ then

$$A \models (I \xrightarrow{f} J) \vee (I \xleftarrow{f} J) \vee (I = J)$$

And so

$$\mathrm{Pts}\,(L) \models (f(I) \xrightarrow{f} f(J)) \vee (f(I) \xleftarrow{f} f(J)) \vee (f(I) = f(J))$$

This implies that $b = d$ so $\psi_{A,L}(f)(I_+) = b = d = \psi_{A,L}(f)(J_+)$ as needed.

To show monotonicity of $\psi_{A,L}(f)$ we get 4 distinct cases. Assuming again that $f(I) = (a, b)$ and $f(J) = (c, d)$:

- If $I_- < J_-$ then we must have

$$A \models (I \xrightarrow{<} J) \vee (I \xrightarrow{m} J) \vee (I \xrightarrow{o} J) \vee (I \xleftarrow{f} J) \vee (I \xleftarrow{d} J)$$

Since f is an $\mathcal{L}_{\mathrm{AIA}}$-embedding, this means that

$$\begin{aligned}\mathrm{Int}\,(L) \models\, &(f(I) \xrightarrow{<} f(J)) \vee (f(I) \xrightarrow{m} f(J)) \vee (f(I) \xrightarrow{o} f(J)) \\ &\vee (f(I) \xleftarrow{f} f(J)) \vee (f(I) \xleftarrow{d} f(J))\end{aligned}$$

So the ordering of the elements a, b, c, d must be one of:

$$a < b < c < d \qquad a < b = c = d \qquad a < c < b < d$$
$$a < c < d = b \qquad a < c < d < b$$

And in all cases $\psi_{A,L}(f)(I_-) = a < c = \psi_{A,L}(f)(J_-)$.

- If $I_- < J_+$ then we must have

$$A \models \neg\,(I \xleftarrow{<} J) \wedge \neg\,(I \xleftarrow{m} J)$$

Since f is an $\mathcal{L}_{\mathrm{AIA}}$-embedding, this means that

$$\mathrm{Int}\,(L) \models \neg\,(f(I) \xleftarrow{<} f(J)) \wedge \neg\,(f(I) \xleftarrow{m} f(J))$$

This is the case with the most orderings of a, b, c, d, having one of:

$$\begin{array}{cccc} a < b < c < d & a < b = c < d & a < c < b < d & a = c < b < d \\ c < a < b = d & c < a < b < d & a = c < b = d & c < a < d < b \end{array}$$
$$a = c < d < b \qquad a < c < d = b \qquad a < c < d < b$$

In all cases though, $\psi_{A,L}(f)(I_-) = a < d = \psi_{A,L}(f)(J_+)$.

- If $I_+ < J_-$ then we must have

$$A \models I \xrightarrow{<} J$$

Since f is an $\mathcal{L}_{\mathrm{AIA}}$-embedding, this means that

$$\mathrm{Int}\,(L) \models f(I) \xrightarrow{<} f(J)$$

So in L we must have $a < b < c < d$, and so $\psi_{A,L}(f)(I_+) = b < c = \psi_{A,L}(f)(J_-)$.

- If $I_+ < J_+$ then we must have

$$A \models (I \xrightarrow{<} J) \vee (I \xrightarrow{m} J) \vee (I \xrightarrow{o} J) \vee (I \xrightarrow{s} J) \vee (I \xrightarrow{d} J)$$

Since f is an $\mathcal{L}_{\mathrm{AIA}}$-embedding, this means that

$$\begin{aligned} \mathrm{Int}\,(L) \models (f(I) \xrightarrow{<} f(J)) \vee (f(I) \xrightarrow{m} f(J)) \vee (f(I) \xrightarrow{o} f(J)) \\ \vee (f(I) \xrightarrow{s} f(J)) \vee (f(I) \xrightarrow{d} f(J) \end{aligned}$$

This gives 5 possible orderings of a, b, c, d in L:

$$a < b < c < d \qquad a < b = c = d \qquad a < c < b < d$$
$$a = c < b < d \qquad c < a < b < d$$

As $b < d$ in all of these, the ordering is preserved by $\psi_{A,L}(f)$.

We expect $\phi_{A,L}$ and $\psi_{A,L}$ to be inverses, which is confirmed by the following:

- Pick any $f : \mathrm{Pts}\,(A) \to L$ and $[(n, I)] \in \mathrm{Pts}\,(A)$, then

$$\begin{aligned} \psi_{A,L}\,(\phi_{A,L}(f))\,([(n, I)]) &= (\text{let } (x_0, x_1) := \phi_{A,L}(f)(I) \text{ in } x_n) \\ &= (\text{let } (x_0, x_1) := (f(I_-), f(I_+)) \text{ in } x_n) \\ &= f([n, I]) \end{aligned}$$

- Pick any $g : A \to \mathrm{Int}(L)$ and $I \in A$, then

$$\phi_{A,L}(\psi_{A,L}(g))(I) = (\psi_{A,L}(g)(I_-), \psi_{A,L}(g)(I_+)) = g(I)$$

Finally, we just have to check naturality of our isomorphism $\phi_{A,L}$. So pick a $\mathcal{L}_{\mathrm{AIA}}$-embedding $f : A \to B$ and a $\mathcal{L}_{\mathrm{SLO}}$-embeddings $g : L \to M$, we need to show the following diagram commutes

$$\begin{tikzcd}
\mathbf{SLO}(\mathrm{Pts}(B), L) \arrow[r, "\mathbf{SLO}(\mathrm{Pts}(f),L)"] \arrow[d, "\phi_{B,L}"'] & \mathbf{SLO}(\mathrm{Pts}(A), L) \arrow[r, "\mathbf{SLO}(\mathrm{Pts}(A),g)"] \arrow[d, "\phi_{A,L}"'] & \mathbf{SLO}(\mathrm{Pts}(A), M) \arrow[d, "\phi_{A,M}"] \\
\mathbf{AIA}(B, \mathrm{Int}(L)) \arrow[r, "\mathbf{AIA}(f,\mathrm{Int}(L))"'] & \mathbf{AIA}(A, \mathrm{Int}(L)) \arrow[r, "\mathbf{AIA}(A,\mathrm{Int}(g))"'] & \mathbf{AIA}(A, \mathrm{Int}(M))
\end{tikzcd}$$

We do this by checking both squares individually:

- **left square**: Pick some $h : \mathbf{SLO}(\mathrm{Pts}(B), L)$ and $I \in A$, then

$$\begin{aligned}
\phi_{A,L}(\mathbf{SLO}(\mathrm{Pts}(f), L)(h))(I) &= \phi_{A,L}(h \circ \mathrm{Pts}(f))(I) \\
&= (h(\mathrm{Pts}(f)(I_-)), h(\mathrm{Pts}(f)(I_+))) \\
&= (h(f(I)_-), h(f(I)_+)) \\
&= \phi_{B,L}(h)(f(I)) \\
&= \phi_{B,L}(h) \circ f(I) \\
&= \mathbf{AIA}(f, \mathrm{Int}(L))(\phi_{B,L}(h))(I)
\end{aligned}$$

- **right square**: Pick some $h : \mathbf{SLO}(\mathrm{Pts}(A), L)$ and $I \in A$, then

$$\begin{aligned}
\phi_{A,M}(\mathbf{SLO}(\mathrm{Pts}(A), g)(h))(I) &= \phi_{A,M}(g \circ h)(I) \\
&= (g(h(I_-)), g(h(I_+))) \\
&= (g(h(I_-)), g(h(I_+))) \\
&= \mathrm{Int}(g)(h(I_-), h(I_+)) \\
&= \mathrm{Int}(g)(\phi_{A,L}(h)(I)) \\
&= \mathbf{AIA}(A, \mathrm{Int}(g))(\phi A, L(h))(I)
\end{aligned}$$

□

In order to get a bit more familiar with these constructions, it can be beneficial to see why $\mathrm{Pts}(-)$ is not right adjoint to $\mathrm{Int}(-)$, that is, in general it is not true

$$\mathbf{AIA}(\mathrm{Int}(L), A) \cong \mathbf{SLO}(L, \mathrm{Pts}(A))$$

for all strict linear orders L and interval algebras A. For example, consider the interval algebra A given by the set $A = \{I, J, K\}$ with the relations between intervals $I \xrightarrow{o} J, J \xrightarrow{m} K, I \xrightarrow{<} K$. Applying the $\mathrm{Pts}(-)$ construction to A gives a linear order with 5 elements, as can be seen in Fig. 1. We will also need the linear order $L = \{1 < 2 < 3\}$, which has 3 intervals, as seen in Fig. 2. Now, an order preserving embedding $f : L \to \mathrm{Pts}(A)$ simply needs to pick 3 distinct points in $\mathrm{Pts}(A)$. There are $\binom{5}{3} = 10$ possible ways of picking 3 distinct points out of $\mathrm{Pts}(A)$, so we see that $|\mathbf{SLO}(L, \mathrm{Pts}(A))| = 10$. On the other hand, an embedding of interval algebras $f : \mathrm{Int}(L) \to A$ must pick out two intervals in A which meet (these will be the images of $(1, 2)$ and $(2, 3)$) with the constraint that the "union" of these two intervals exists in A. In this specific case, our only option is that $f((1, 2)) = J$ and $f((2, 3)) = K$, but there is no interval which is started by J and finished by K, so there is nowhere to map $(1, 3)$ to. This means that $\mathbf{AIA}(\mathrm{Int}(L), A) = \emptyset$, so there can be no isomorphism between this and $\mathbf{SLO}(L, \mathrm{Pts}(A))$.

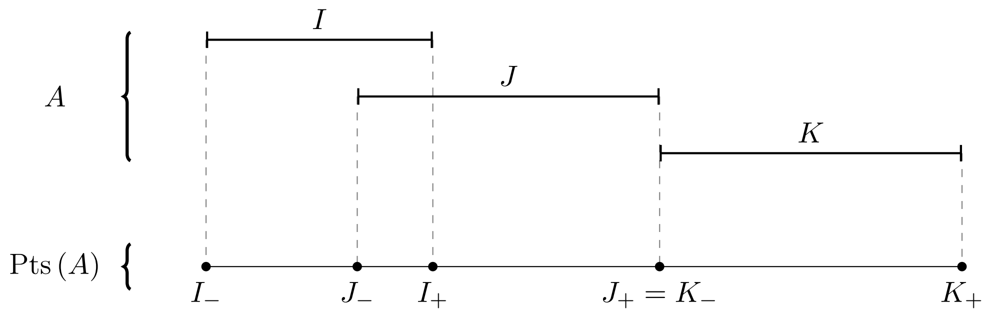


Figure 1: An example of the $\mathrm{Pts}(-)$ construction.

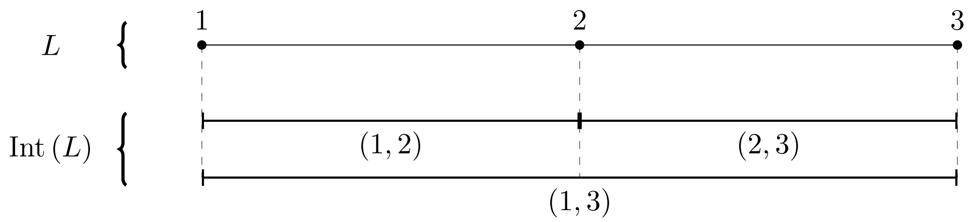


Figure 2: An example of the $\mathrm{Int}(-)$ construction

4.1 Characterising the Unit and Counit

Given an adjunction, it is always helpful to compute the associated unit and counit natural transformations. In our case, the unit is a natural transformation $\eta : \mathrm{id}_{\mathbf{AIA}} \to \mathrm{Int}(\mathrm{Pts}(-))$ whose component at an interval algebra A is given by $\eta_A = \phi_{A,\mathrm{Pts}(A)}(\mathrm{id}_{\mathrm{Pts}(A)})$. Computing this, we get the map

$$\eta_A : A \to \mathrm{Int}(\mathrm{Pts}(A)) \text{ sending } I \mapsto (I_-, I_+)$$

Recall that each component of the unit η_A is an arrow in **AIA**, so it must be an embedding. This means that every interval algebra A can be seen as a substructure of the intervals $\mathrm{Int}\,(L)$ for some linear order L, namely for $L = \mathrm{Pts}\,(A)$.

Dually, the counit here is a natural transformation $\epsilon : \mathrm{Pts}\,(\mathrm{Int}\,(-)) \to \mathrm{id}_{\mathbf{SLO}}$ where each of its components is given by $\epsilon_L = \psi_{\mathrm{Int}(L),L}(\mathrm{id}_{\mathrm{Int}(L)})$. Fixing some L and working through this construction, we get

$$\epsilon_L : \mathrm{Pts}\,(\mathrm{Int}\,(L)) \to L \text{ sending } (a, b)_- \mapsto a \text{ and } (a, b)_+ \mapsto b$$

Proposition 4.5. *The counit component at a strict linear order L, ϵ_L, is an isomorphism if and only if $|L| \neq 1$.*

Proof. First, suppose that $|L| = 1$, so $L = \{a\}$. Then $\mathrm{Int}\,(L) = \emptyset$ as we do not allow empty intervals and so $\mathrm{Pts}\,(\mathrm{Int}\,(L)) = \emptyset$ too. As such, ϵ_L cannot be an isomorphism.

For the converse, notice that ϵ_L is a map in **SLO**, hence it must be injective. Provided that $|L| \neq 1$ then it also turns out to be surjective: fix some $a \in L$, there must be at least one distinct $b \in L$. Now either $a < b$, so (a, b) is a valid interval over L and then $\epsilon_L((a, b)_-) = a$. Alternatively, $b < a$, so (b, a) is an interval in $\mathrm{Int}\,(L)$ and $\epsilon_L((b, a)_+) = a$ $\square$

As for the unit, it will give us an isomorphism when our interval algebra already has all possible intervals. More precisely, we say that an interval algebra has all possible intervals if it satisfies the following sentence

$$\begin{aligned}
\phi_{\text{full}} = &\Big(\forall I,\ \forall J,\ (I \xrightarrow{<} J) \to \exists K,\ (I \xrightarrow{m} K) \wedge (K \xrightarrow{m} J)\Big)\\
&\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{m} J) \to \exists K,\ (I \xleftarrow{s} K) \wedge (K \xrightarrow{f} J)\Big)\\
&\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{o} J) \to \exists K,\ (I \xleftarrow{f} K) \wedge (K \xrightarrow{s} J)\Big)\\
&\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{s} J) \to \exists K,\ (I \xrightarrow{m} K) \wedge (K \xrightarrow{f} J)\Big)\\
&\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{f} J) \to \exists K,\ (I \xleftarrow{m} K) \wedge (K \xrightarrow{s} J)\Big)\\
&\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{d} J) \to \exists K,\ (I \xleftarrow{m} K) \wedge (K \xrightarrow{s} J)\Big)
\end{aligned}$$

The above formula does not specify it, but it can be shown that the K being quantified over must actually be unique. For example, fix some I and J such that $I \xrightarrow{<} J$ and let there be two K_1, K_2 such that $I \xrightarrow{m} K_i \xrightarrow{m} J$. Then we have $K_1 \xleftarrow{m} I \xrightarrow{m} K_2$, implying that $K_1 \xrightarrow{s=S} K_2$. If $K_1 \xrightarrow{s} K_2$ then $K_1 \xrightarrow{s} K_2 \xrightarrow{m} J$ and our axioms mean that $K_1 \xrightarrow{<} J$, which is a contradiction as the interval algebra relations are mutually exclusive. Similarly we can show that $K_1 \xleftarrow{s} K_2$ cannot happen, leaving us with the only option of $K_1 = K_2$. Similar arguments show the uniqueness of K in the other cases.

Proposition 4.6. *The unit component at an interval algebra A, η_A, is an isomorphism if and only if $A \models \phi_{full}$.*

Proof. First, suppose that η_A is an isomorphism. To show that $A \models$ *allints* we fix two intervals $I, J \in A$. Now we have to consider 6 cases:

- If $I \xrightarrow{<} J$ then in $\mathrm{Int}\,(\mathrm{Pts}\,(A))$ we must have $I_- < I_+ < J_- < J_+$. This means the following are all valid intervals

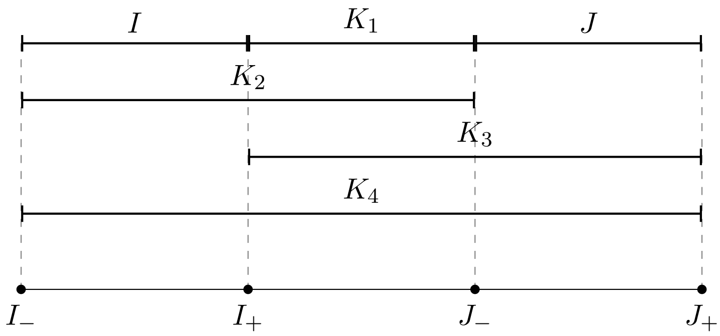
$$\eta_A(I) = (I_-, I_+) \xrightarrow{m} (I_+, J_-) \xrightarrow{m} (J_-, J_+) = \eta_A(J)$$

  so taking $K = \eta_A^{-1}((I_+, J_-))$ we see that the first conjunct holds.
- If $I \xrightarrow{m} J$ then $I_- < I_+ = J_- < J_+$, hence taking $K = \eta_A^{-1}((I_-, J_+))$ shows that the second conjunct holds.
- If $I \xrightarrow{o} J$ then $I_- < J_- < I_+ < J_+$ and we can take $K = \eta_A^{-1}((J_-, I_+))$ to prove the third conjunct.
- If $I \xrightarrow{s} J$ then $I_- = J_- < I_+ < J_+$, so taking $K = \eta_A^{-1}((I_+, J_+))$ proves the fourth conjunct.
- If $I \xrightarrow{f} J$ then $J_- < I_- < I_+ = J_+$ so the choice $K = \eta_A^{-1}((J_-, I_-))$ gives a proof of the penultimate conjunct.
- If $I \xrightarrow{d} J$ then $J_- < I_- < I_+ < J_+$ and if we let $K = \eta_A^{-1}((J_-, I_-))$ then we see the last conjunct holds too.

As all conjuncts hold, this implies that $A \models \phi_{\text{full}}$ as expected.

Next, suppose that we have an interval algebra A such that $A \models \phi_{\text{full}}$ and fix two intervals $I, J \in A$. Since I and J are arbitrary, it will suffice to consider the basic non-dual relations.

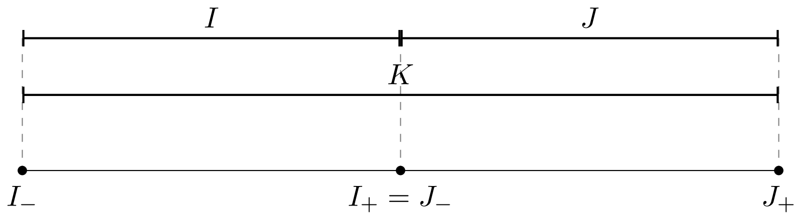
- If $I \xrightarrow{<} J$ then we get some K_1 which is met by I and meets J. Then we can glue I and K_1 to get K_2 and similarly we glue K_1 and J to get K_3. Finally, gluing K_2 and J gives us K_4. This gives the following picture



Then the unit η_A surjects on all intervals over the start and end points of I, J:

$$(I_-, J_-) = \eta_A(K_2) \qquad (I_-, J_+) = \eta_A(K_4)$$
$$(I_+, J_-) = \eta_A(K_1) \qquad (I_+, J_+) = \eta_A(K_3)$$

- If $I \xrightarrow{m} J$ then we can glue I and J to get K, yielding the following

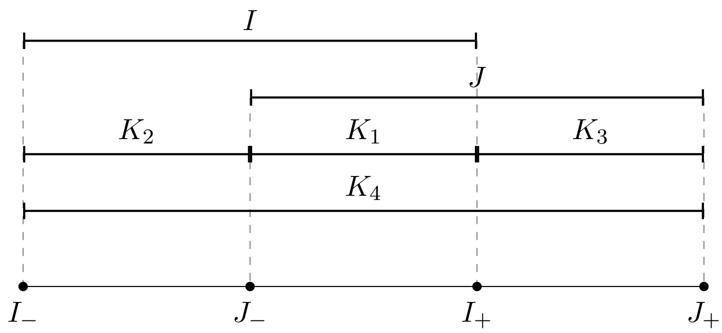


The unit also surjects on all valid intervals over the start and end poits of I, J:

$$(I_-, J_-) = \eta_A(J) \qquad (I_-, J_+) = \eta_A(K) \qquad (I_+, J_+) = \eta_A(J)$$

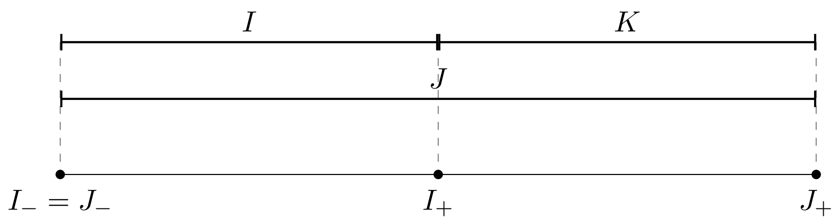
In this case there are only 3 in this case since $I_+ = J_-$ so (I_+, J_-) is not a valid interval over $\mathrm{Pts}(A)$.

- If $I \xrightarrow{o} J$ then first we intersect I and J giving K_1. Similarly we can intersect K_1 with I and J giving K_2 and K_3 respectively. Finally gluing I and K_3 together gives K_4.



Similar to the $I \xrightarrow{<} J$ case, the unit is then surjective onto the 4 possible intervals involving I_-, I_+, J_-, J_+.

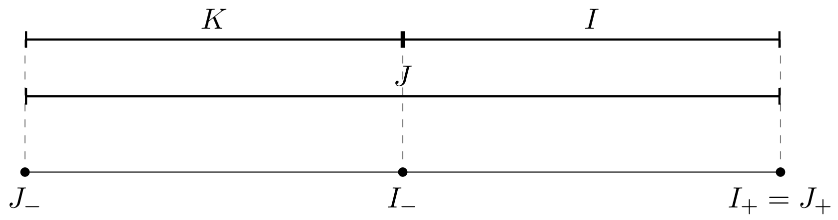
- When $I \xrightarrow{s} J$ we can intersec I and J to get K



We get 4 intervals over the start and end points of I and J, which are given by

$$(I_-, J_+) = \eta_A(J) \qquad (I_-, I_+) = \eta_A(I) \qquad (I_+, J_+) = \eta_A(K)$$

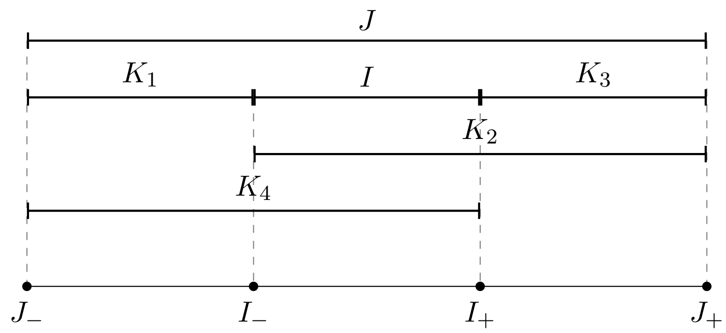
- The $I \xrightarrow{f} J$ follows similarly to the previous case, by intersecting I and J to get K



And again the 3 valid intervals over I_-, I_+, J_-, J_+ all lie in the image of the unit

$$(J_-, I_-) = \eta_A(K) \qquad (J_-, I_+) = \eta_A(J) \qquad (I_-, J_+) = \eta_A(I)$$

- Finally there is the $I \xrightarrow{d} J$ case. By ϕ_{full} we can get the initial segment K_1 of J which meets I. Then we intersect K_1 and J to get K_2, which in turn we intersect with I to get K_3. Finally gluing K_1 and I gives K_4.



This gives 4 intervals which all lie in the image of the unit

$$\begin{aligned} (J_-, I_-) &= \eta_A(K_1) \qquad & (J_-, I_+) &= \eta_A(K_4) \\ (I_-, J_+) &= \eta_A(K_2) \qquad & (I_+, J_+) &= \eta_A(K_3) \end{aligned}$$

This is sufficient to see that η_A is surjective. After all fix some arbitrary interval

$$([(n, I)], [(m, J)]) \in \operatorname{Int}(\operatorname{Pts}(A))$$

If $I = J$ then we must have

$$([(n, I)], [(m, J)]) = (I_-, I_+) = \eta_A(I)$$

If $I \neq J$, then after possibly swapping I, J we may assume $I \xrightarrow{<mosfd} J$, which brings us to one of the cases handled above, where all intervals involing I_-, I_+, J_- and J_+ were in the image of the unit. □

Corollary 4.7. *There exists an equivalence of categories between the full subcategory of strict linear orders L with $|L| \neq 1$ and the full subcategory of interval algebras satisfying ϕ_{full}.*

Proof. An adjunction can always be restricted to an equivalence of categories by considering the full subcategories where the unit and counit are isomorphisms. [14] □

5 The Model Theory of Interval Algebras

5.1 The Fraïssé Class of Finite Interval Algebras

We start by considering the class **FIA** of finite interval algebras. From our work in Section 2.2.1, we know that **FIA** satisfies the HP, since $\mathbb{T}_{\text{AIA}}$ is universal and relational. $\mathcal{L}_{\text{AIA}}$ is also finite, so **FIA** must be EC.

To prove that **FIA** has the JEP and AP, notice that $\text{Pts}(-)$ must send finite interval algebras to finite strict linear orders. In fact, given an interval algebra A, $|\text{Pts}(A)| \leq 2|A|$ since $\text{Pts}(A)$ is a quotient of $A + A$. Similarly, $\text{Int}(-)$ must also send finite strict linear orders to finite interval algebras as given a strict linear order L, $|\text{Int}(L)| = \binom{|L|}{2}$. Using this fact, we will be able to reduce the proof of these properties to the proof that **FCh** satisfies them.

Proposition 5.1. *The class* ***FIA*** *has the joint embedding property.*

Proof. Given two finite interval algebras A and B, we use the JEP of strict linear orders to get the following diagram in **SLO**

$$\begin{tikzcd}
\text{Pts}(A) \arrow[dr, "f"] & \\
& \Omega \\
\text{Pts}(B) \arrow[ur, "g"'] &
\end{tikzcd}$$

Then, applying $\text{Int}(-)$ and using the adjunction unit η, we get

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] & \text{Int}(\text{Pts}(A)) \arrow[dr, "\text{Int}(f)"] & \\
& & \text{Int}(\Omega) \\
B \arrow[r, "\eta_B"'] & \text{Int}(\text{Pts}(B)) \arrow[ur, "\text{Int}(g)"'] &
\end{tikzcd}$$

And the composites $\text{Int}(f) \circ \eta_A$ and $\text{Int}(g) \circ \eta_B$ along with the interval algebra Ω give us the joint embedding of A and B. $\square$

Proposition 5.2. *The class* ***FIA*** *has the amalgamation property.*

Proof. Suppose we have the following diagram in **AIA**

$$\begin{tikzcd}
& A \\
C \arrow[ur, "f"] \arrow[dr, "g"'] & \\
& B
\end{tikzcd}$$

Applying $\mathrm{Pts}\,(-)$ takes us to **SLO**, at which point we can use the AP of strict linear orders to get the commuting square

$$\begin{tikzcd}
& \mathrm{Pts}\,(A) \arrow[dr, "f'"] & \\
\mathrm{Pts}\,(C) \arrow[ur, "\mathrm{Pts}(f)"] \arrow[dr, "\mathrm{Pts}(g)"'] & & \Omega \\
& \mathrm{Pts}\,(B) \arrow[ur, "g'"'] &
\end{tikzcd}$$

Now going back to **AIA** gives the commuting diagram

$$\begin{tikzcd}
& A \arrow[r, "\eta_A"] & \mathrm{Int}\,(\mathrm{Pts}\,(A)) \arrow[dr, "\mathrm{Int}(f')"] & \\
C \arrow[ur, "f"] \arrow[r, "\eta_C"] \arrow[dr, "g"'] & \mathrm{Int}\,(\mathrm{Pts}\,(C)) \arrow[ur, "\mathrm{Int}(\mathrm{Pts}(f))"] \arrow[dr, "\mathrm{Int}(\mathrm{Pts}(g))"'] & & \mathrm{Int}\,(\Omega) \\
& B \arrow[r, "\eta_B"'] & \mathrm{Int}\,(\mathrm{Pts}\,(B)) \arrow[ur, "\mathrm{Int}(g')"'] &
\end{tikzcd}$$

For the AP of the finite interval algebras, we are only interested in the outer square. The necessary maps are then $\mathrm{Int}\,(f') \circ \eta_A$ and $\mathrm{Int}\,(g') \circ \eta_B$, both mapping into $\mathrm{Int}\,(\Omega)$. □

Theorem 5.3. *The Fraïssé limit of **FIA** is Int* $(\mathbb{Q})$

Proof. To show that the Fraïssé limit of the finite interval algebras is $\mathrm{Int}\,(\mathbb{Q})$, it suffices to show that $\mathrm{Age}(\mathrm{Int}\,(\mathbb{Q})) = \mathbf{FIA}$ and that $\mathrm{Int}\,(\mathbb{Q})$ is homogeneous.

To see that $\mathrm{Age}(\mathrm{Int}\,(\mathbb{Q})) = \mathbf{FIA}$, prove that both sides include into the other. Suppose we have some finitely generated $\mathcal{L}_{\mathrm{AIA}}$-substructure $A \subset \mathrm{Int}\,(\mathbb{Q})$. Since $\mathbb{T}_{\mathrm{AIA}}$ is universal, A must also be an interval algebra. Furthermore, since $\mathcal{L}_{\mathrm{AIA}}$ is relational, A must also be finite, so $A \in \mathbf{FIA}$. For the converse inclusion, consider some finite interval algebra A, it must embed into $\mathrm{Int}\,(\mathrm{Pts}\,(A))$, which in turn embeds into $\mathrm{Int}\,(\mathbb{Q})$ (since $\mathrm{Pts}\,(A)$ is finite, hence embeddable into $\mathbb{Q}$). Restricting these composition of these embeddings onto their image in $\mathrm{Int}\,(\mathbb{Q})$ then gives the needed isomorphism.

As for why $\operatorname{Int}(\mathbb{Q})$ is homogeneous, fix two $\mathcal{L}_{\mathrm{AIA}}$-substructures $A, B \subseteq \operatorname{Int}(\mathbb{Q})$, along with some $\mathcal{L}_{\mathrm{AIA}}$-isomorphism $f : A \to B$. In essence we have the following diagram of interval algebras, where i and j are the inclusions into $\operatorname{Int}(\mathbb{Q})$:

$$\begin{tikzcd}
\operatorname{Int}(\mathbb{Q}) & \operatorname{Int}(\mathbb{Q}) \\
A \arrow[u, "i"] \arrow[r, "f"] & B \arrow[u, "j"']
\end{tikzcd}$$

Applying $\operatorname{Int}(-)$ to move to linear orders, we can postcompose $\operatorname{Pts}(i)$ and $\operatorname{Pts}(j)$ with the counit at $\mathbb{Q}$ to realise $\operatorname{Pts}(A)$ and $\operatorname{Pts}(B)$ as $\mathcal{L}_{\mathrm{SLO}}$-substructures of $\mathbb{Q}$. Now, $\operatorname{Pts}(f)$ is still an isomorphism as these are preserved by functors, and using the fact that $\mathbb{Q}$ is homogeneous, we can extend $\operatorname{Pts}(f)$ to an isomorphism g, giving the commuting diagram

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, "g"] & \mathbb{Q} \\
\operatorname{Pts}(\operatorname{Int}(\mathbb{Q})) \arrow[u, "\epsilon_{\mathbb{Q}}"] & \operatorname{Pts}(\operatorname{Int}(\mathbb{Q})) \arrow[u, "\epsilon_{\mathbb{Q}}"'] \\
\operatorname{Pts}(A) \arrow[u, "\operatorname{Pts}(i)"] \arrow[r, "\operatorname{Pts}(f)"] & \operatorname{Pts}(B) \arrow[u, "\operatorname{Pts}(j)"']
\end{tikzcd}$$

Finally, we apply $\operatorname{Int}(-)$ to bring us back to interval algebras, where we have the following diagram:

$$\begin{tikzcd}
 & \operatorname{Int}(\mathbb{Q}) \arrow[r, "\operatorname{Int}(g)"] & \operatorname{Int}(\mathbb{Q}) & \\
\operatorname{Int}(\mathbb{Q}) \arrow[ur, bend left, "\mathrm{id}_{\operatorname{Int}(\mathbb{Q})}"] \arrow[r, "\eta_{\operatorname{Int}(\mathbb{Q})}"'] & \operatorname{Int}(\operatorname{Pts}(\operatorname{Int}(\mathbb{Q}))) \arrow[u, "\operatorname{Int}(\epsilon_{\mathbb{Q}})"'] & \operatorname{Int}(\operatorname{Pts}(\operatorname{Int}(\mathbb{Q}))) \arrow[u, "\operatorname{Int}(\epsilon_{\mathbb{Q}})"] & \operatorname{Int}(\mathbb{Q}) \arrow[ul, bend right, "\mathrm{id}_{\operatorname{Int}(\mathbb{Q})}"'] \arrow[l, "\eta_{\operatorname{Int}(\mathbb{Q})}"] \\
A \arrow[u, "i"] \arrow[r, "\eta_A"'] & \operatorname{Int}(\operatorname{Pts}(A)) \arrow[u, "\operatorname{Int}(\operatorname{Pts}(i))"'] \arrow[r, "\operatorname{Int}(\operatorname{Pts}(f))"] & \operatorname{Int}(\operatorname{Pts}(B)) \arrow[u, "\operatorname{Int}(\operatorname{Pts}(i))"] & B \arrow[u, "j"'] \arrow[l, "\eta_B"] \\
 & A \arrow[ul, bend left, "\mathrm{id}_A"] \arrow[u, "\eta_A"'] \arrow[r, "f"'] & B \arrow[ur, bend right, "\mathrm{id}_B"'] \arrow[u, "\eta_B"] &
\end{tikzcd}$$

Although not obvious at first, the above diagram commutes, to check this we look at all the "irreducible components" individually:

- The middle rectangle in red commutes since it already commuted for linear orders.
- The top triangles commute by the triangle identities of our adjunction.
- The squares to the left, right and bottom of the red rectangle commute by naturality of the unit.
- The bottom triangles commute since the identity is the unit of composition.

Chasing around the outside of the diagram, we see that

$$\mathrm{Int}\,(g) \circ \mathrm{id}_{\mathrm{Int}(\mathbb{Q})} \circ i \circ \mathrm{id}_A = \mathrm{id}_{\mathrm{Int}(\mathbb{Q})} \circ j \circ \mathrm{id}_B \circ f$$

Simplifying shows that $\mathrm{Int}\,(g) \circ i = j \circ f$. Now, since g was an isomorphism, so is $\mathrm{Int}\,(g)$, so we have successfully extended f to an automorphism of $\mathrm{Int}\,(\mathbb{Q})$. $\square$

5.2 Stable and NIP Interval Agebras

The question of stability for interval algebras is answered similarly to linear orders:

Theorem 5.4. *An interval algebra A is stable if and only if it is finite.*

Proof. Suppose that A is an interval algebra and consider the formula

$$\phi_{\mathrm{lex}}(I, J) = (I \xrightarrow{<} J) \vee (I \xrightarrow{m} J) \vee (I \xrightarrow{o} J) \vee (I \xrightarrow{s} J) \vee (I \xleftarrow{f} J) \vee (I \xleftarrow{d} J)$$

The above formula takes the elements of A and lexicographically orders them, by first comparing the start times of each interval, and then the end times if the start times coincide. As a result, the formula ϕ_{lex} gives the structure of a strict linear order to the elements of A:

- **irreflexivity**: Fix some interval $I \in A$, then $I = I$. Since the relational symbols of interval algebras along with equality are all mutually exclusive, neither $I \xrightarrow{i} I$ nor $I \xleftarrow{i} I$ can hold for any $i \in \{<, m, o, s, f, d\}$. This means $A \models \neg\phi_{\mathrm{lex}}(I, I)$ so ϕ_{lex} is irreflexive.
- **transitivity**: Fix three intervals $I, J, K \in A$ and suppose that $A \models \phi_{\mathrm{lex}}(I, J)$ and $A \models \phi_{\mathrm{lex}}(J, K)$. Since ϕ_{lex} consists of the disjunction of 6 relational symbols, there are 36 cases which would lead to this situation. Considering each of these cases individually and looking up our transitivity axioms for interval algebras, we can find all possible relations between I and K, which is detailed in Table 12. In all of these cases, we must still have $A \models \phi_{\mathrm{lex}}(I, K)$, so ϕ_{lex} is transitive.
- **trichotomy**: Fix two intervals $I, J \in A$. First notice that $A \models \phi_{\mathrm{lex}}(J, I)$ if and only if we have

$$A \models (I \xleftarrow{<} J) \vee (I \xleftarrow{m} J) \vee (I \xleftarrow{o} J) \vee (I \xleftarrow{s} J) \vee (I \xrightarrow{f} J) \vee (I \xrightarrow{d} J)$$

which is the same as taking the dual of all the relation symbols in $\phi_{\text{lex}}(I, J)$. Hence $A \models \phi_{\text{lex}}(I, J) \vee (I = J) \vee \phi_{\text{lex}}(J, I)$ is equivalent to saying that the interval algebra relations are exhaustive, which is one of our axioms. Hence the strict ordering given by ϕ_{lex} is linear.

$I \longrightarrow J$	$J \longrightarrow K$	$I \longrightarrow K$	$I \longrightarrow J$	$J \longrightarrow K$	$I \longrightarrow K$
$<$	$<$	$<$	s	$<$	$<$
$<$	m	$<$	s	m	$<$
$<$	o	$<$	s	o	$< mo$
$<$	s	$<$	s	s	s
$<$	F	$<$	s	F	$< mo$
$<$	D	$<$	s	D	$< moFD$
m	$<$	$<$	F	$<$	$<$
m	m	$<$	F	m	m
m	o	$<$	F	o	o
m	s	m	F	s	o
m	F	$<$	F	F	F
m	D	$<$	F	D	D
o	$<$	$<$	D	$<$	$< moFD$
o	m	$<$	D	m	oFD
o	o	$< mo$	D	o	oFD
o	s	o	D	s	oFD
o	F	$< mo$	D	F	D
o	D	$< moFD$	D	D	D

Table 12: Cases for transitivy of ϕ_{lex}.

Now for any finite interval algebra A, all strict linear orders interpretable in A will be bounded in size by $|A|^k$ for some $k \in \mathbb{N}$. In particular, all such linear orders must be finite, so A is stable.

For the converse, suppose A is stable. Using ϕ_{lex} we can linearly order all the elements of A, but as A is stable this linear order must be finite, hence A must be finite. □

Restricting our attention to interval algebras with all possible intervals, we see they all have the NIP, much like their underlying linear orders.

Theorem 5.5. *For any linear order L, the interval algebra $\mathrm{Int}(L)$ has the NIP.*

Proof. Suppose that $\mathrm{Int}\,(L)$ has the IP, so there is some formula $\phi(x; y)$ which has the IP. Let the IP of $\phi(x; y)$ be realised by the model $A \models \mathrm{Th}(\mathrm{Int}\,(L))$ and sequences $(a_i)_{i<\omega}$ and $(b_I)_{I\subseteq\omega}$. Since finite interval algebras are stable and hence NIP, we may assume that L is infinite, so the counit ϵ_L is an isomorphism. By the triangle laws of our adjunction we know that

$$\mathrm{Int}\,(\epsilon_L) \circ \eta_{\mathrm{Int}(L)} = \mathrm{id}_{\mathrm{Int}(L)}$$

As ϵ_L is an isomorphism, $\mathrm{Int}\,(\epsilon_L)$ must also be an isomorphism, so $\eta_{\mathrm{Int}(L)}$ too is an isomorphism. In particular, this implies that $\mathrm{Int}\,(L) \models \phi_{\mathrm{full}}$. As a result, $A \models \phi_{\mathrm{full}}$ too and $A \cong \mathrm{Int}\,(L')$ for some strict linear order L'. Recall that $\mathrm{Int}\,(L')$ is interpretable in L', hence we can translate the formula $\phi(x; y)$ to some formula $\phi'(x'; y')$ in the language of strict linear orders. In addition, we can also translate the sequences of tuples $(a_i)_{i<\omega}$ and $(b_I)_{I\subseteq\omega}$ to sequences of tuples in L'. This means that the formula $\phi'(x'; y')$ has the IP, making it so the linear order L' also has the IP. This is a contradiction as all linear orders have the NIP. □

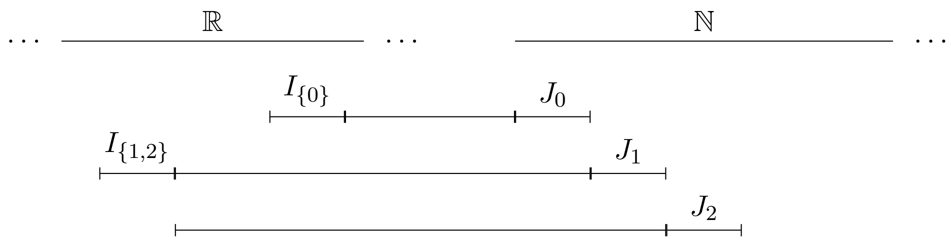
However, as we saw before with the exponential fields case, by modifying models even slightly we can add enough expressive power to get the IP. In our case, by starting with an interval algebra with the NIP, we can remove enough intervals to allow for the IP.

Example 5.6. Let $L = \mathbb{R} \sqcup \mathbb{N}$ be the linear order given by putting all the elements of $\mathbb{R}$ before $\mathbb{N}$. By picking out the right substructure $A \subsetneq \mathrm{Int}\,(L)$ we should get an interval algebra with the IP.

First fix some bijection $f : \mathcal{P}(\mathbb{N}) \xrightarrow{\sim} \mathbb{R}$ and then let A be the substructure of $\mathrm{Int}\,(L)$ given by

$$\begin{aligned} A = &\big\{(x, y) \;\big|\; x, y \in \mathbb{R} \text{ such that } x < y\big\} \\ &\cup \big\{(f(I), i) \;\big|\; I \subseteq \mathbb{N}, i \in I\big\} \\ &\cup \big\{(x, y) \;\big|\; x, y \in \mathbb{N} \text{ such that } x < y\big\} \end{aligned}$$

In particular, the only intervals $(x, a) \in A$ with $x \in \mathbb{R}$ and $a \in \mathbb{N}$ come from the set comprehension $\big\{(f(I), i) \;\big|\; I \subseteq \mathbb{N}, i \in I\big\}$. Using I_S to denote the interval $(f(S) - 1, f(S)$ and J_n to denote $(n, n + 1)$, we would end up with something like



Now the formula

$$\phi(I; J) = \exists K,\ (J \xrightarrow{m} K) \wedge (K \xrightarrow{m} I)$$

has the IP and this is realised in A by the sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_I)_{I\subseteq\mathbb{N}}$ defined by

$$a_i = (i, i + 1) \qquad\qquad b_I = (f(I) - 1, f(I))$$

Since for any $i \in \mathbb{N}$ and $I \subseteq \mathbb{N}$, $A \models \phi(a_i, b_I)$ if and only if there exists some interval $(x, y) \in A$ with $x, y \in L$ such that

$$(f(I) - 1, f(I)) \xrightarrow{m} (x, y) \wedge (x, y) \xrightarrow{m} (i, i + 1)$$

This can only happen if $x = f(I)$ and $y = i$, so we would require

$$(f(I), i) \in \left\{(f(I), i) \;\middle|\; I \subseteq \mathbb{N}, i \in I\right\}$$

which happens if and only if $i \in I$ as needed.

6 Evaluation

At the start of this project we had one overarching objective, that was to come up with some axioms for the theory of interval algebras and study their model theoretic consequences. As such the main topic of concern for our evaluation will be deciding how appropriate our axiomatisation was.

Firstly, we should consider the models allowed by our axiomatisation. We saw that the models of $\mathbb{T}_{\text{AIA}}$ coincide with the class of substructures of the interval algebras $\text{Int}(L)$. These are exactly the type of models that underlie the work in [5], on this front we should be very happy with our axiomatisation. In addition, we also saw that $\mathbb{T}_{\text{AIA}} \cup \{\phi_{\text{full}}\}$ axiomatised the class of models of the form $\text{Int}(L)$, letting us distinguish an important class of models through a single sentence.

In addition, $\mathbb{T}_{\text{AIA}}$ is also finite, meaning there is very little ambiguity when checking whether a structure is a model or not. On the other hand, despite being finite, there are quite a lot of axioms in $\mathbb{T}_{\text{AIA}}$ due to its many relations. This means proving things about interval algebras can be quite labourious, with a lot of quite distinct cases to check. If we could find some symmetry in the transitivity relations this would help a lot, but there does not seem to be anything too tangible to simplify reasoning.

Finally, with our given axioms, we managed to prove all of the results we were originally interested at the start of this project, so our axiomatisation was definitely something we could work with.

7 Conclusion

We started this report by establishing a list of axioms for the theory of interval algebras, which we showed to be satisfiable, containing at least the main class of models we expected. Then, through further study of the interval construction, we saw it to be the right adjoint of the points construction. The unit of this adjunction let us see that all models were a substructure of the intervals over an appropriate linear order, cementing our axiomatisation as worth considering.

Through this adjunction, we managed to reduce the most of the work on the Fraïssé limit of **FIA** to the proofs for **FCh**, giving a very elegant and conceptual proof that the limit of **FIA** was $\mathrm{Int}(\mathbb{Q})$.

By ordering the intervals in a linear order lexicographically, we saw that the stable and finite interval algebras coincided and in attempting to characterise the interval algebras for which the adjunction unit was an isomorphism, we found a formula ϕ_{full} axiomatising the interval algebras of the form $\mathrm{Int}(L)$. Since the interval algebra $\mathrm{Int}(L)$ is always interpretable in L, this let us conclude that also had the NIP.

This is where our work finished, leaving some possibilities for future work.

First, suppose that we removed the $I \xrightarrow{<} J$ conjunct from ϕ_{full} to get

$$\begin{aligned}\phi_{\text{fullish}} = &\Big(\forall I,\ \forall J,\ (I \xrightarrow{m} J) \to \exists K,\ (I \xleftarrow{s} K) \wedge (K \xrightarrow{f} J)\Big)\\ &\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{o} J) \to \exists K,\ (I \xleftarrow{f} K) \wedge (K \xrightarrow{s} J)\Big)\\ &\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{s} J) \to \exists K,\ (I \xrightarrow{m} K) \wedge (K \xrightarrow{f} J)\Big)\\ &\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{f} J) \to \exists K,\ (I \xleftarrow{m} K) \wedge (K \xrightarrow{s} J)\Big)\\ &\wedge \Big(\forall I,\ \forall J,\ (I \xrightarrow{d} J) \to \exists K,\ (I \xleftarrow{m} K) \wedge (K \xrightarrow{s} J)\Big)\end{aligned}$$

Then it would be interesting to see the type of interval algebras satisfying ϕ_{fullish}. We expect such an interval algebra A to look like a gluing of $\{\mathrm{Int}(L_i)\}_{i\in I}$ where the indexing set I is linearly ordered. If this were true, then letting L be the linear order cosntructed by gluing $\{L_i\}_{i\in I}$, we would expect A to be interpretable in the linear order L plus some colouring predicates. A linear order always has the NIP, regardless of the number of colouring predicates added, so this would let us extend our result about NIP interval algebras.

Another question possibly worth considering further comes from our work done with the interval construction, which we saw to be a right adjoint functor. From category theory we know that right adjoints preserve limits, and dually that left adjoints preserve colimits. Work by Caramello [15] shows how to realise the Fraïssé limit of a Fraïssé class as a colimit in the language of category theory. So here we have a functor, $\mathrm{Int}(-)$, preserving quite a complicated colimit. This could hint at a couple of options: it might be the case that

Int $(-)$ is also a left adjoint functor, although if this is the case it would have to be some new functor distinct from Pts $(-)$. Alternatively, there could be something specific about Fraïssé limits which meant they should be preserved by right adjoints. In either case, it would be interesting to see what develops.

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